Knowledge Compilation for Boolean Functional Synthesis

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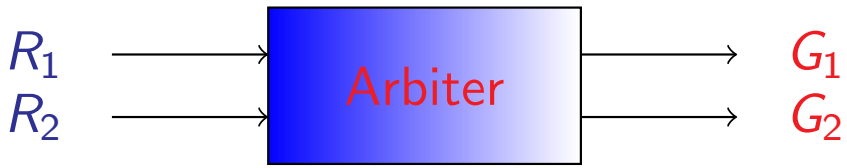

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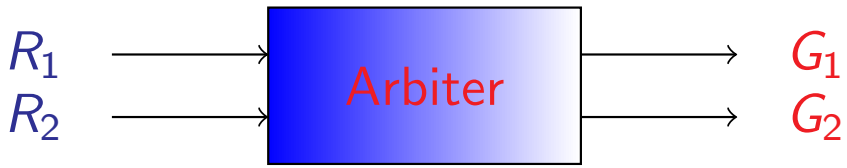

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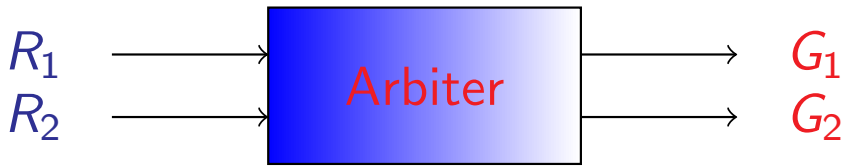

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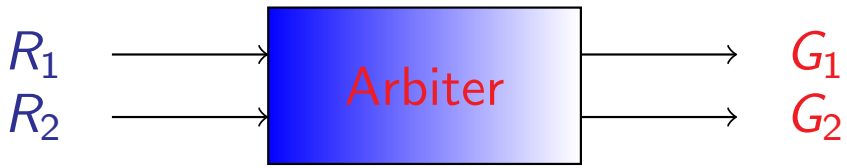

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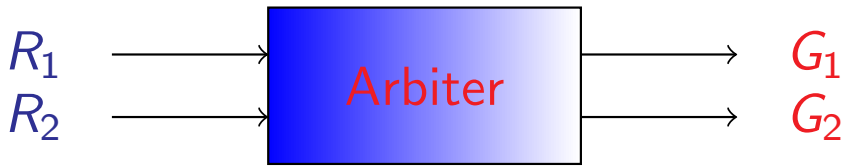

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Boolean Functional Synthesis (BFnS)

Synthesizing Boolean functions from a relational specification.

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Formal definition

Given Boolean relation $\varphi(x_1, .., x_n, y_1, .., y_m)$

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Synthesize Boolean functions $F_j(x_1 \ldots x_n)$ for each y_j s.t.

$\forall x_1 \ldots x_n \, (\exists y_1 \ldots y_m \; \varphi(x_1 \ldots x_n, y_1 \ldots y_m) \;\Leftrightarrow\; \varphi(x_1, \ldots x_n, F_1, \ldots F_m))$

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Interesting even if for some $x_1, \ldots x_n$, there isn't $y_1, \ldots y_m$ s.t. $\varphi(x_1 \ldots x_n, y_1 \ldots y_m)$ holds.

Applications of Boolean Functional Synthesis

1. Factorization: Interesting but hard for synthesis!
2. Disjunctive decomposition of transition relations [Trivedi et al'02]
3. Quantifier elimination, of course!
   - $\exists Y\ \varphi(X, Y) \equiv \varphi(X, F(X))$
4. Certifying QBF-SAT solvers [Bendetti'05, ...]
5. (Reactive) circuit/strategy synthesis [Alur et al'05, ...]
6. Program synthesis
   - Combinatorial sketching [Solar-Lezama et al'06, Srivastava et al'13, ...]
   - Complete functional synthesis [Kuncak et al'10, ...]
7. Repair/partial synthesis of circuits [Fujita et al'13 ...]

Existing Approaches

1. Closely related to Boolean unification
   1. Given $\varphi(\mathbf{X}, y_1, .., y_n) = 0$, find most general unifiers of $y_1, .., y_m$ in terms of $\mathbf{X}$.
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2. Extract Skolem functions from proof of validity of $\forall \mathbf{X} \exists \mathbf{Y} \varphi(\mathbf{X}, \mathbf{Y})$
   - Bendetti'05, Jussilla et al'07, Jiang et al'11, Heule et al'14, ...
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3. Generate Skolem functions matching given templates
   - Srivastava et al'13, ...
   - Effective when small set of candidate Skolem functions known
   - Not always the case

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7. Quantifier instantiation techniques in SMT solvers
   - Barrett et al'15, Bierre et al'17
   - Works even for bit-vector and other theories
   - May not scale well for large specs with large # quantified vars
8. Input-output separation: Tabajara et al'18
9. Initial approximation iterative refinement Akshay et al'16

How Hard (or Easy) Is It?

- BFnS is *NP*-hard
  - Consider a co-NP^{NP} (or Π_2^P)-complete problem:
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- Lower bound results in circuit-size refer to monotone circuits [Razbarov 1985; Alon and Boppana 1987]
  - Skolem functions need not be monotone
  - Different argument for lower bounds on Skolem circuits

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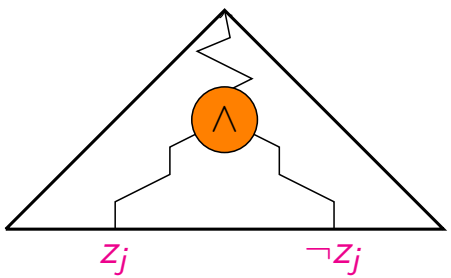

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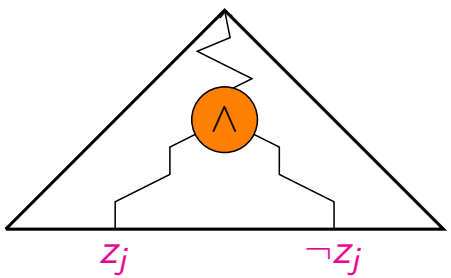

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Experiments: Efficient synthesis even when F not in wDNNF

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- **YES:** Super-polynomial time in worst-case
- **Practical performance promising!**

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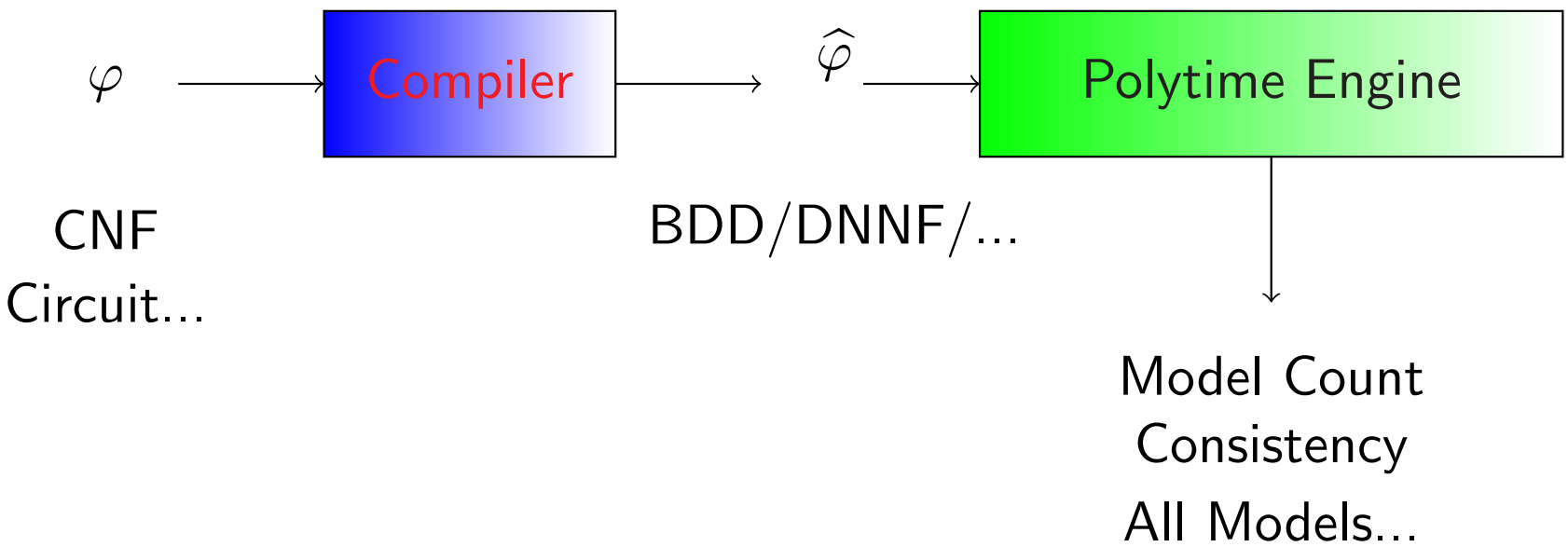

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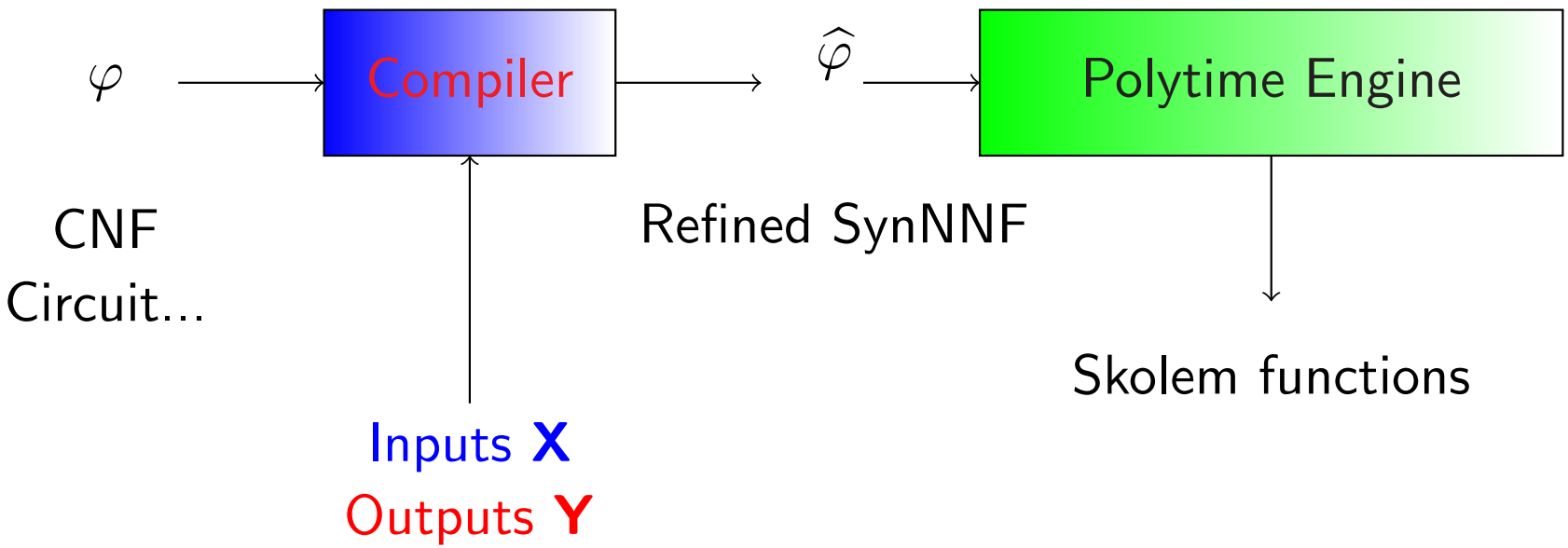

SynNNF: A Quick Introduction

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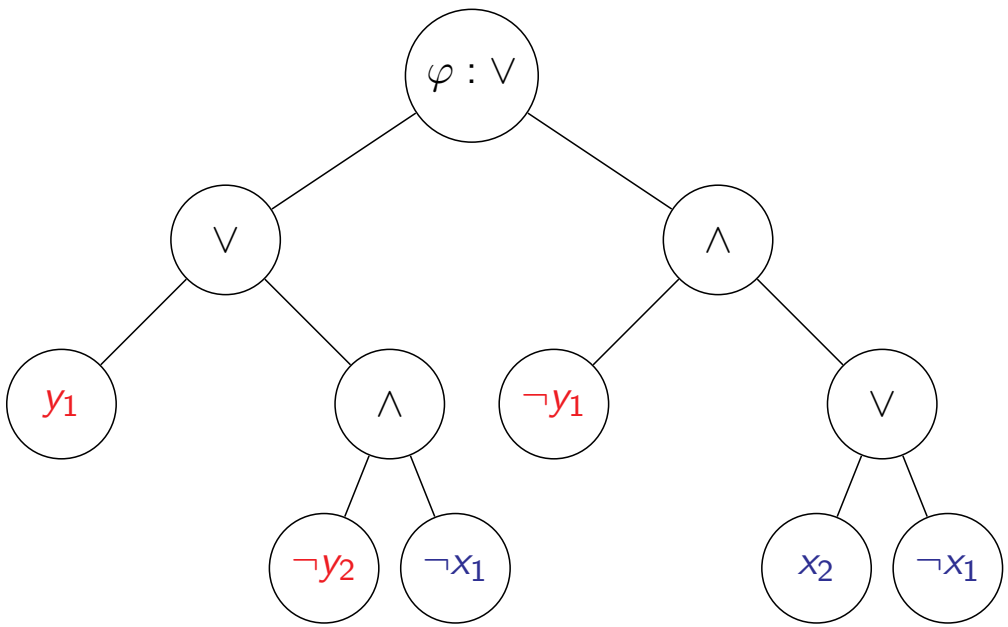

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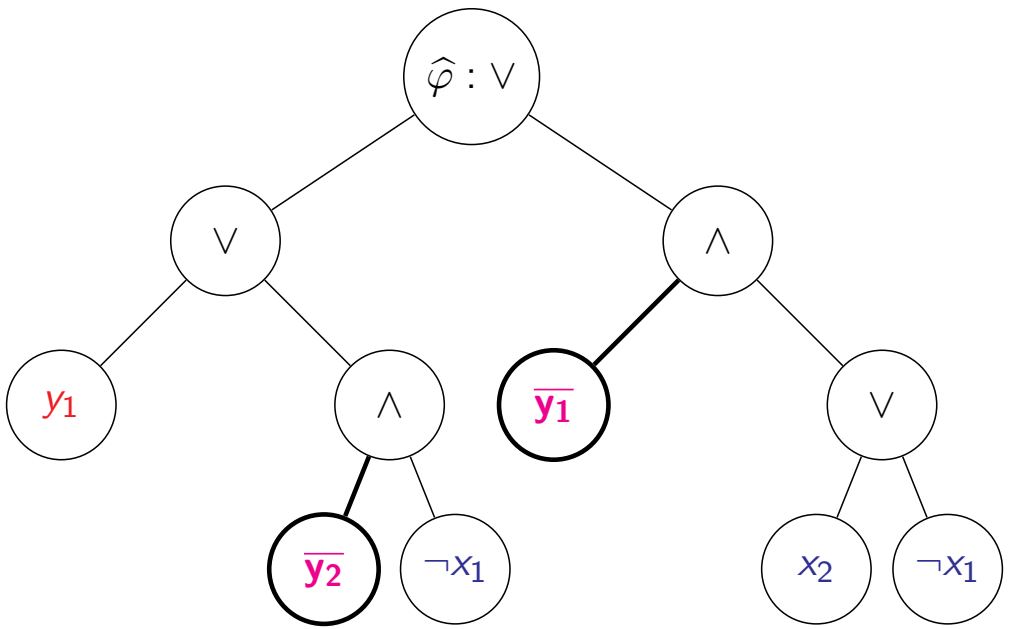

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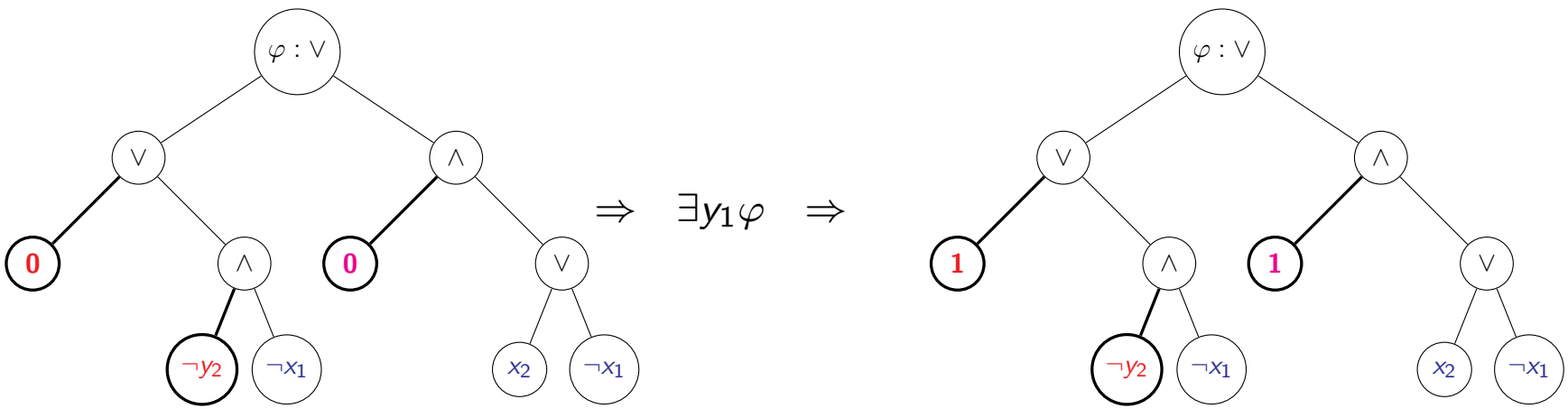

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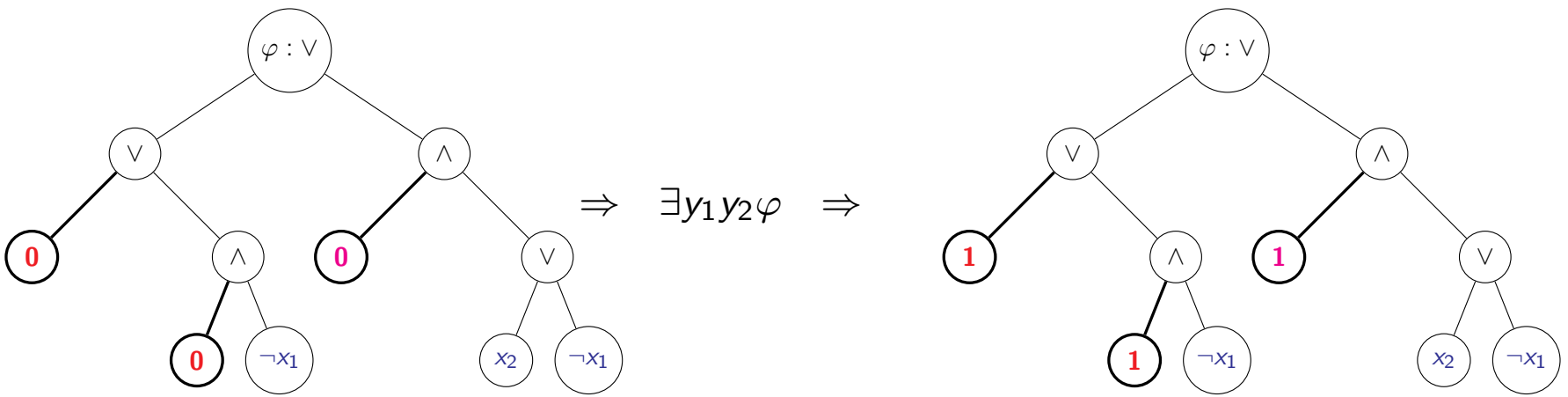
- $\widehat{\varphi}(x_1 \ldots x_n, \overbrace{1..1}^{i}, y_{i+1} \ldots y_m, \overbrace{1..1}^{i}, \neg y_{i+1} \ldots \neg y_m) \Leftarrow \exists y_1 \ldots y_i \ \varphi(\ldots)$



SynNNF: A Quick Introduction

Given

- Positive form of spec $\widehat{\varphi}(x_1, \ldots x_n, y_1, \ldots y_m, \overline{y_1}, \ldots \overline{y_m})$
- Sequence (linear order) of outputs $(y_1, \ldots y_m)$

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Representation of φ is in **SynNNF**

SynNNF: Some observations

- Every propositional formula representable in SynNNF for every ordering of outputs
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SynNNF: Some observations

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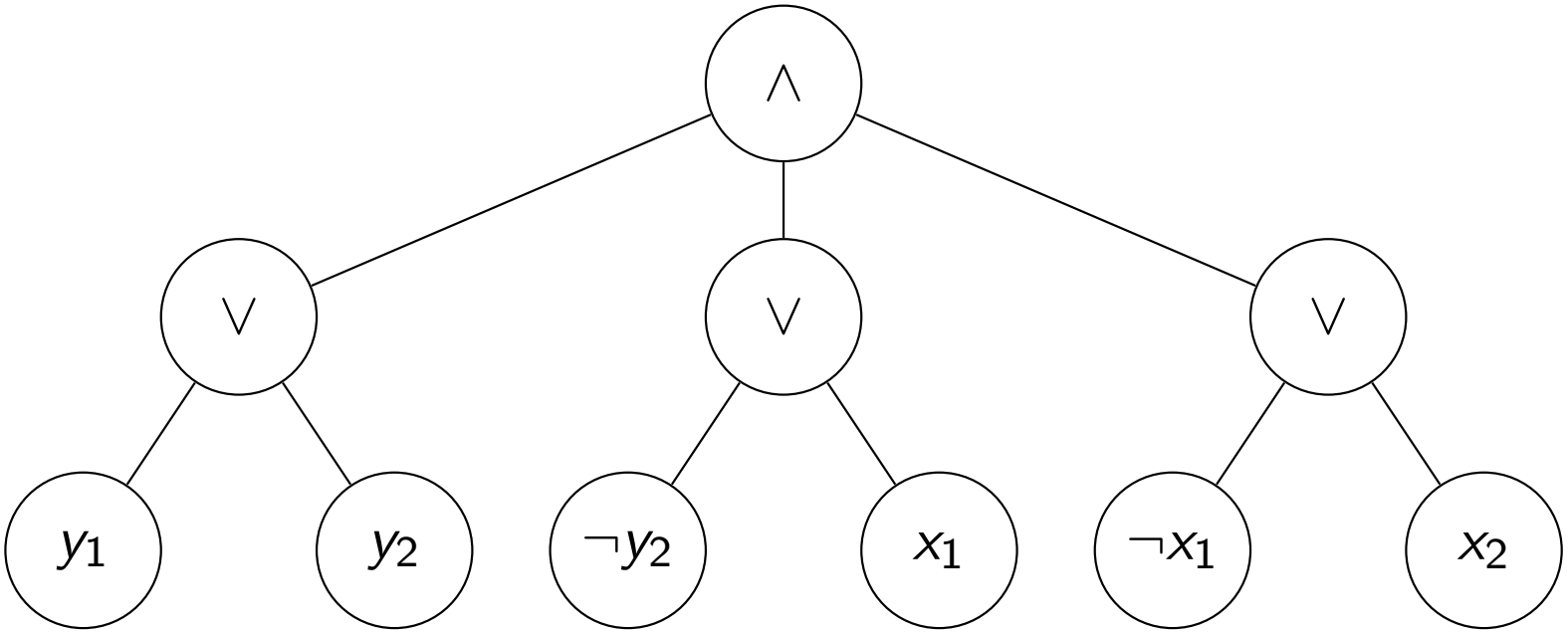
  - DNF always SynNNF for any output order
- A formula may have multiple SynNNF representations
  - DNF, BDD, DNNF ...
- A given representation may be SynNNF for order1 of outputs and not in SynNNF for order2 of outputs.
  - Example coming up ...

SynNNF: An Example

$$\varphi(x_1, x_2, y_1, y_2) = (y_1 \vee y_2) \wedge (\neg y_2 \vee x_1) \wedge (\neg x_1 \vee x_2)$$



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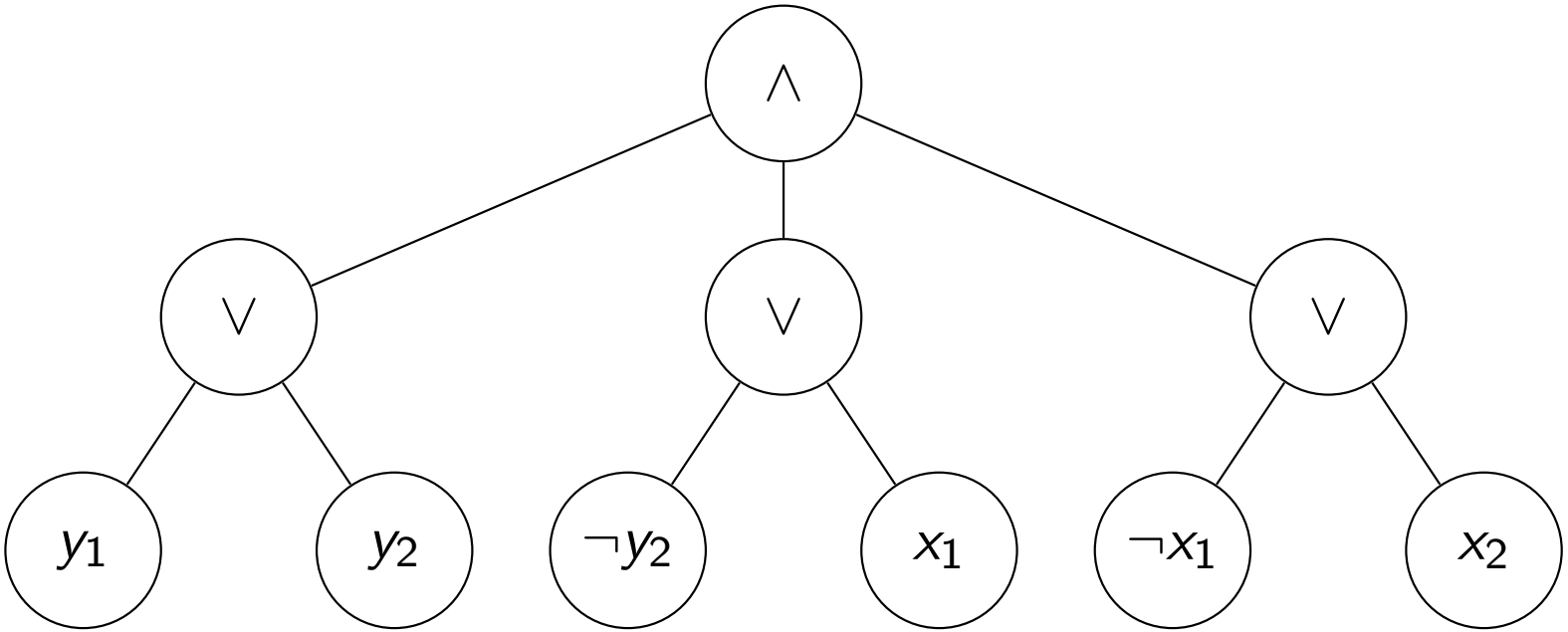
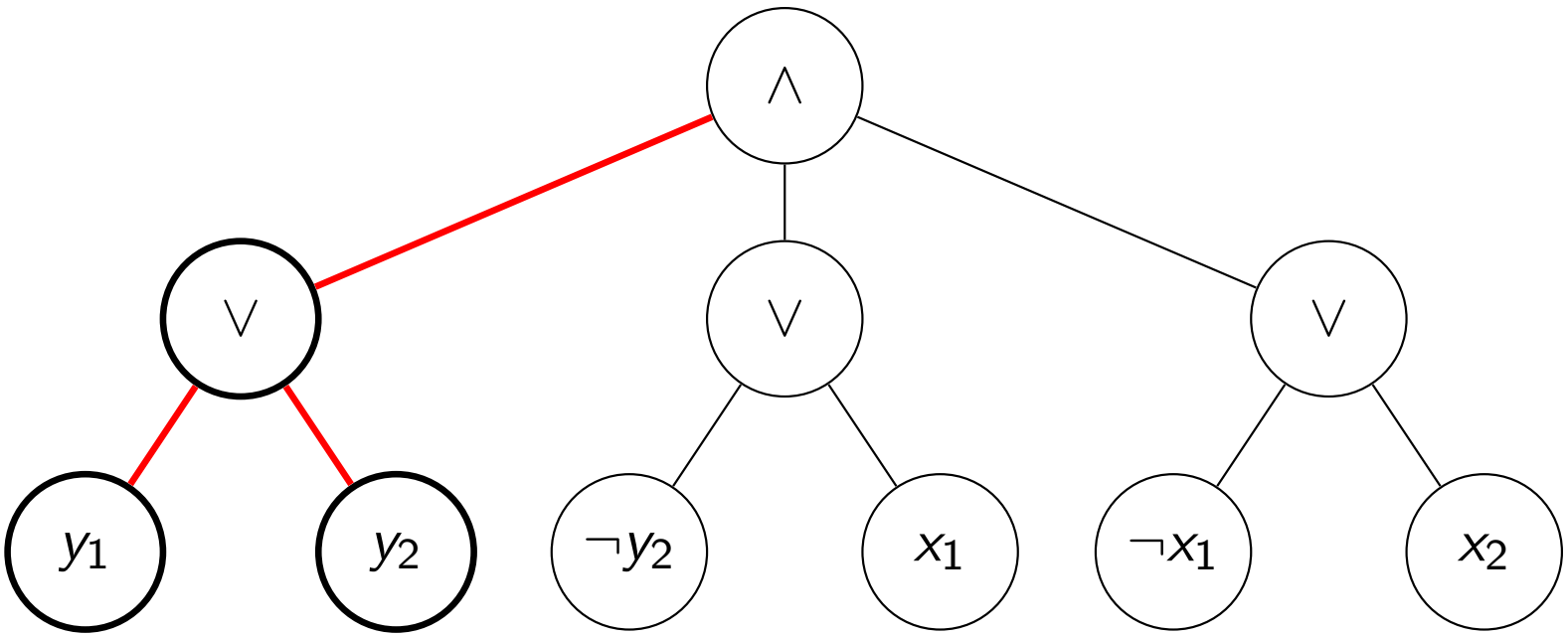


Representation of φ **not** in DNNF/wDNNF

SynNNF: An Example

$\varphi(x_1, x_2, y_1, y_2) = (y_1 \vee y_2) \wedge (\neg y_2 \vee x_1) \wedge (\neg x_1 \vee x_2)$

Output sequence: $(\underline{y_1}, y_2)$



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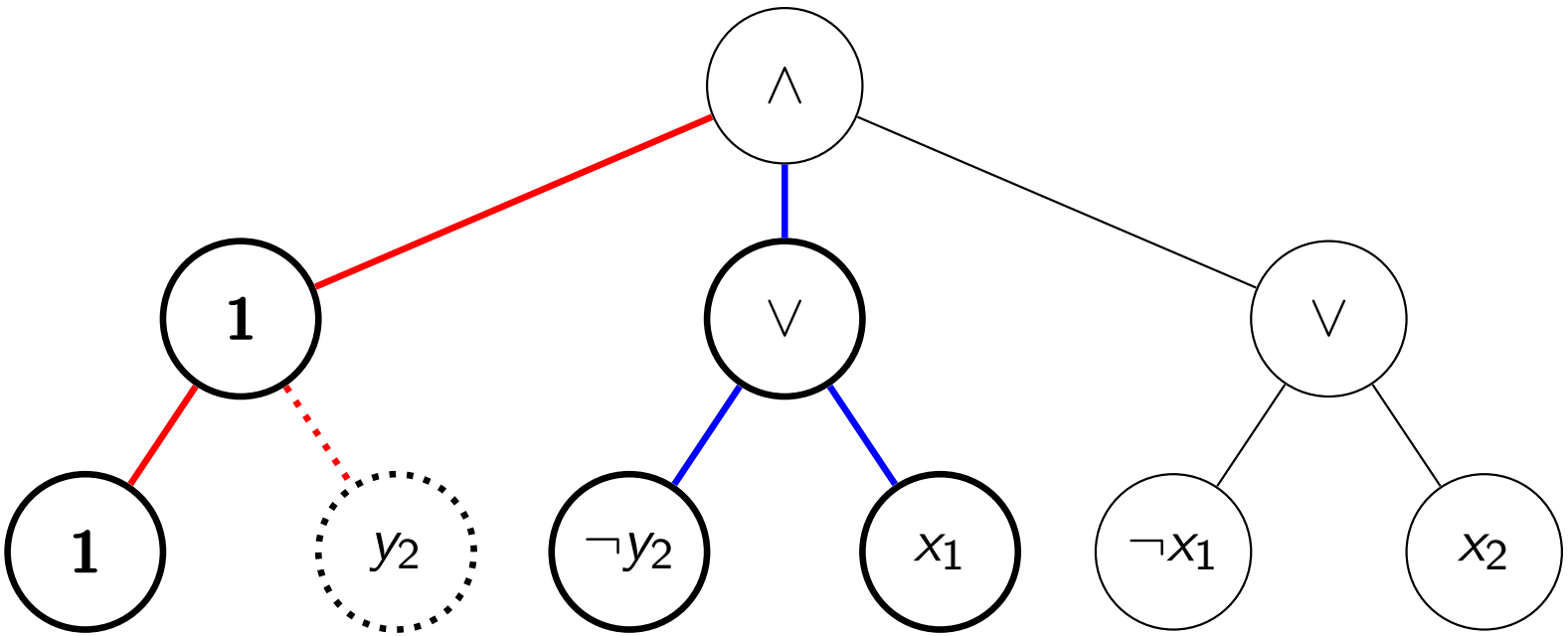

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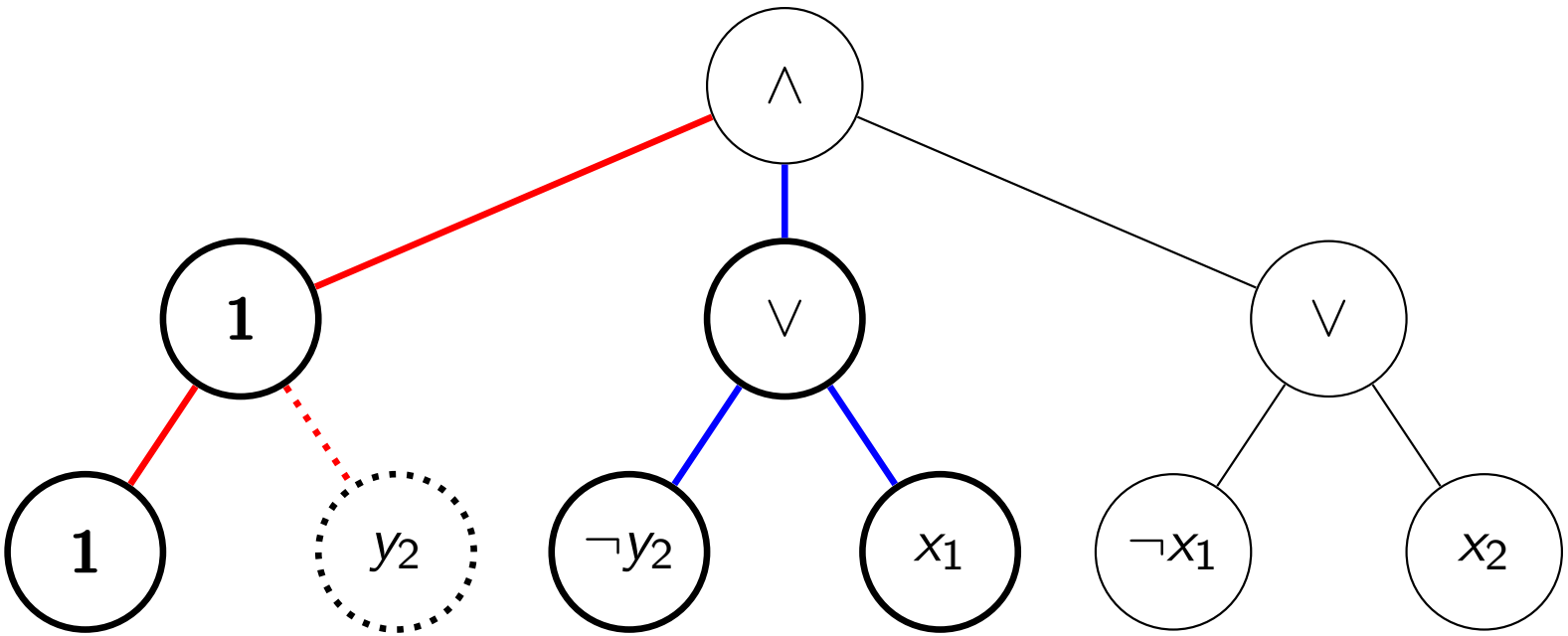
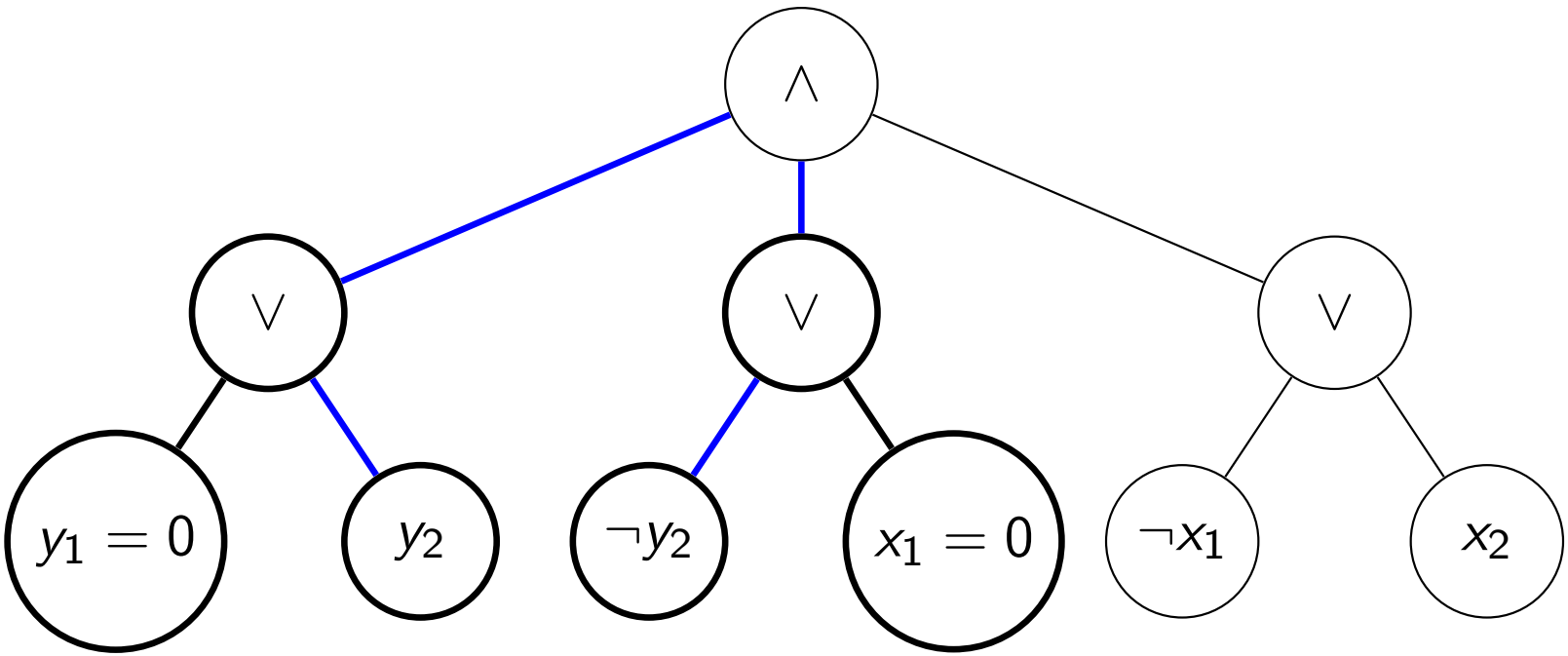


Representation of φ in SynNNF **w.r.t** (y_1, y_2)

Non-SynNNF: An Example

$\varphi(x_1, x_2, y_1, y_2) = (y_1 \vee y_2) \wedge (\neg y_2 \vee x_1) \wedge (\neg x_1 \vee x_2)$

Output sequence: $(\underline{y_2}, y_1)$



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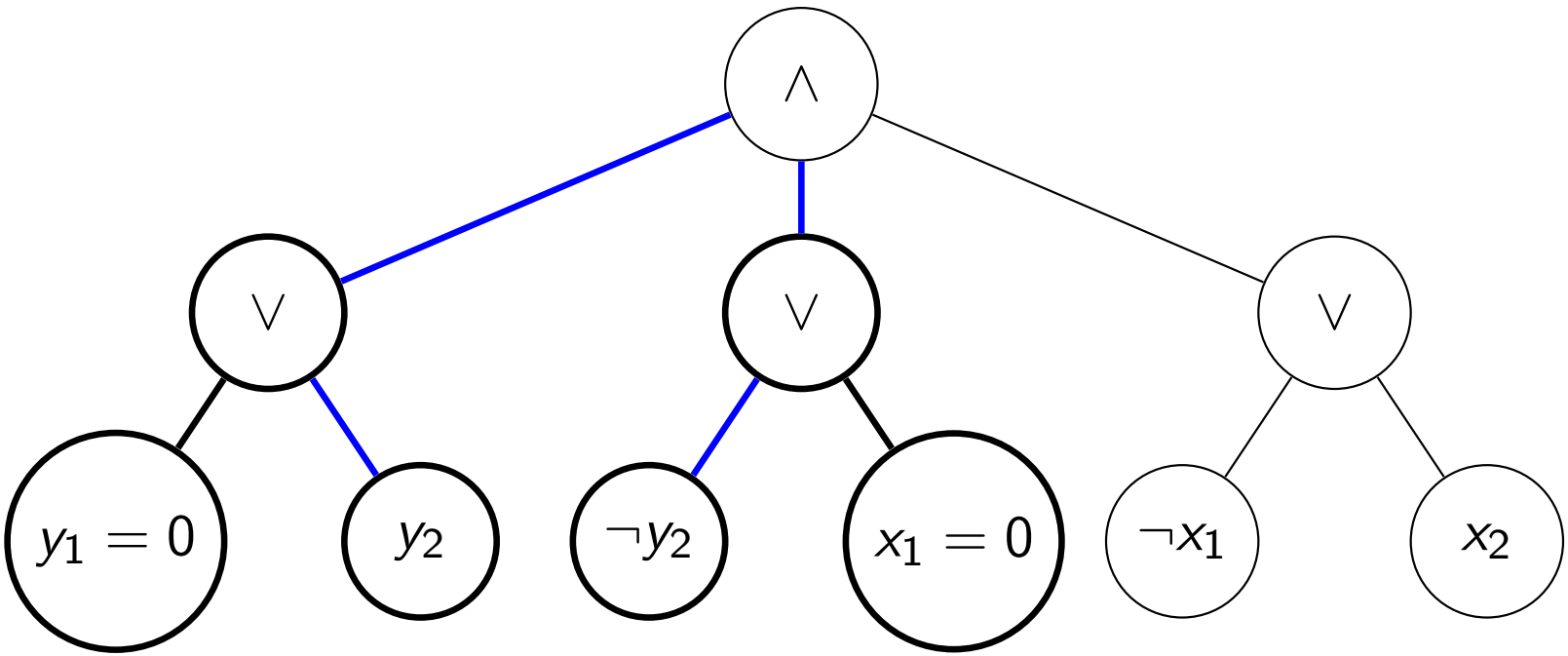
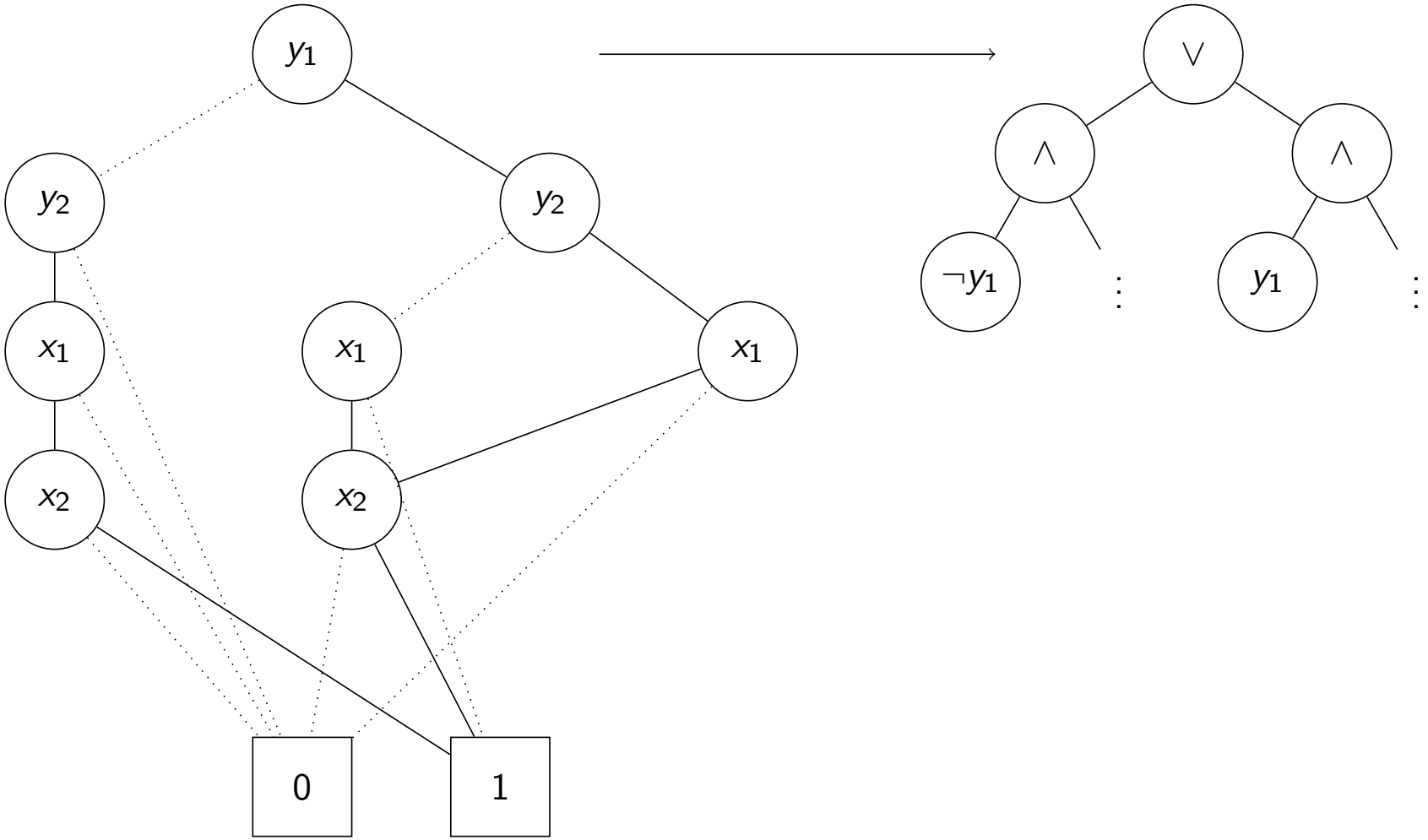
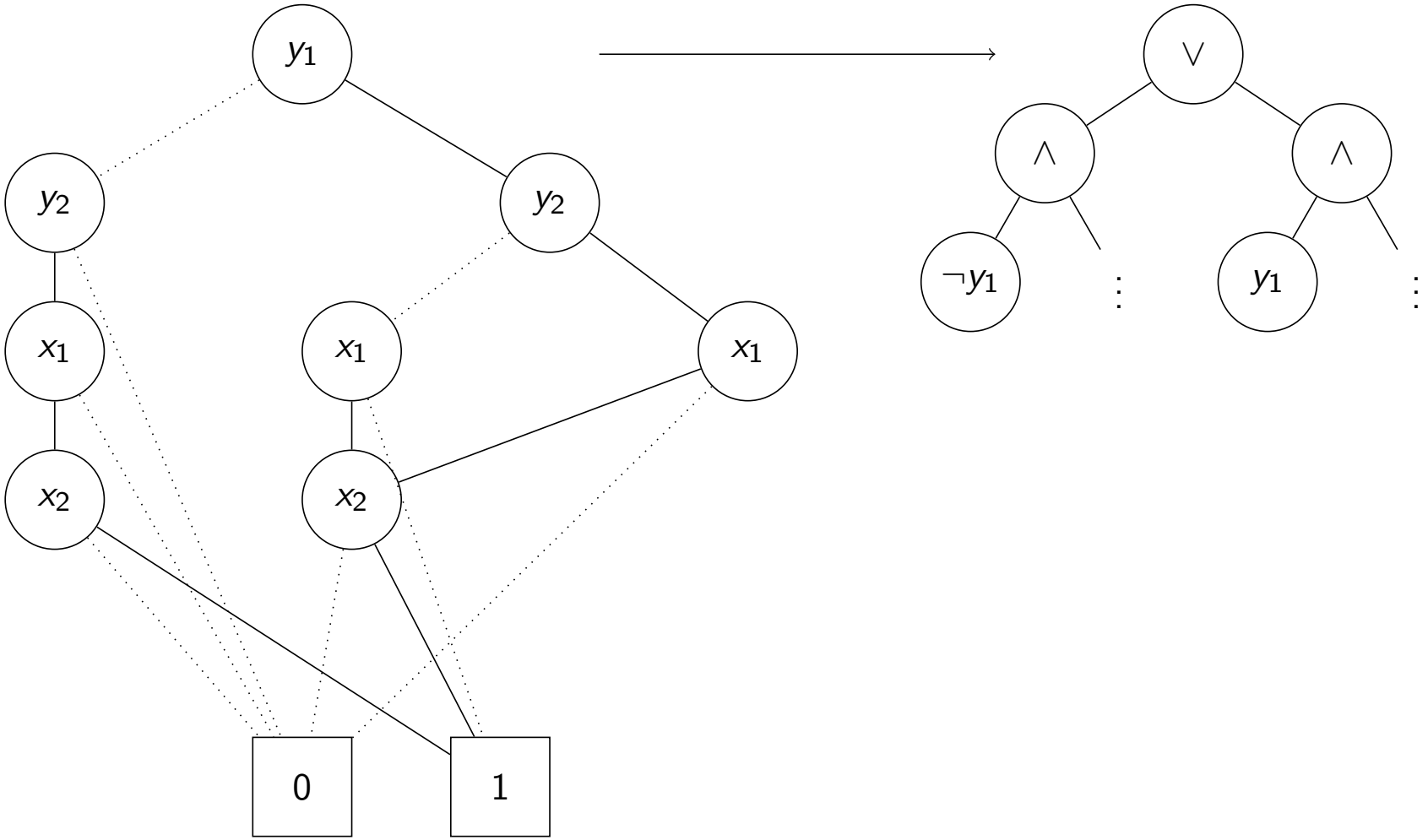


Representation of φ not in SynNNF **w.r.t** (y_2, y_1)

BDD and SynNNF



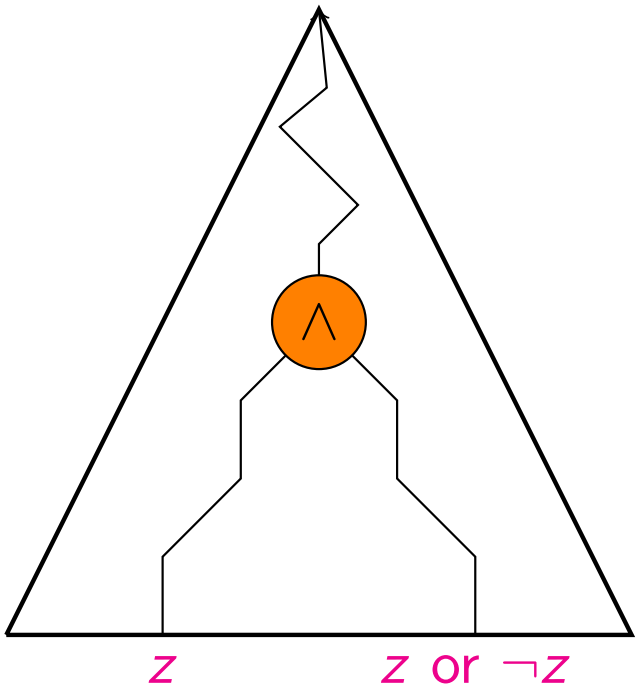
BDD and SynNNF



BDD $\rightarrow$ SynNNF in linear time for any output sequence and any BDD variable ordering.

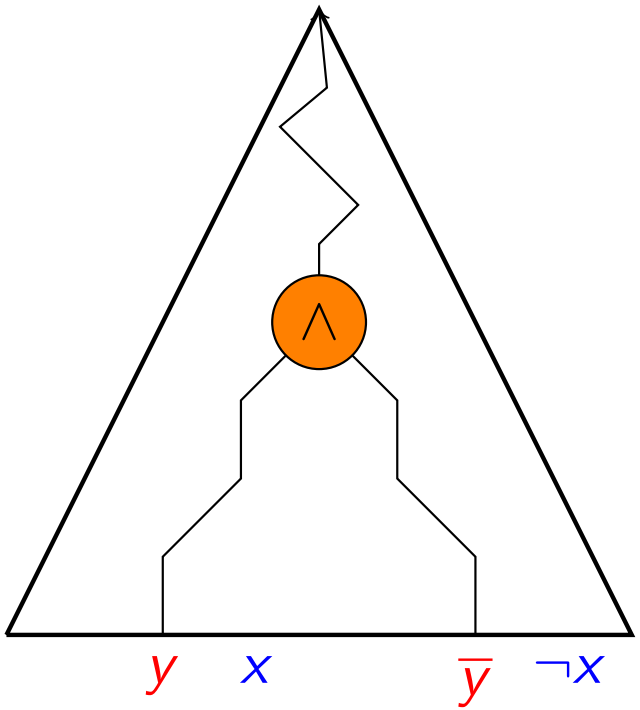
DNNF, wDNNF and SynNNF

φ in DNNF



Disallowed paths

φ in wDNNF



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DNNF, wDNNF and SynNNF

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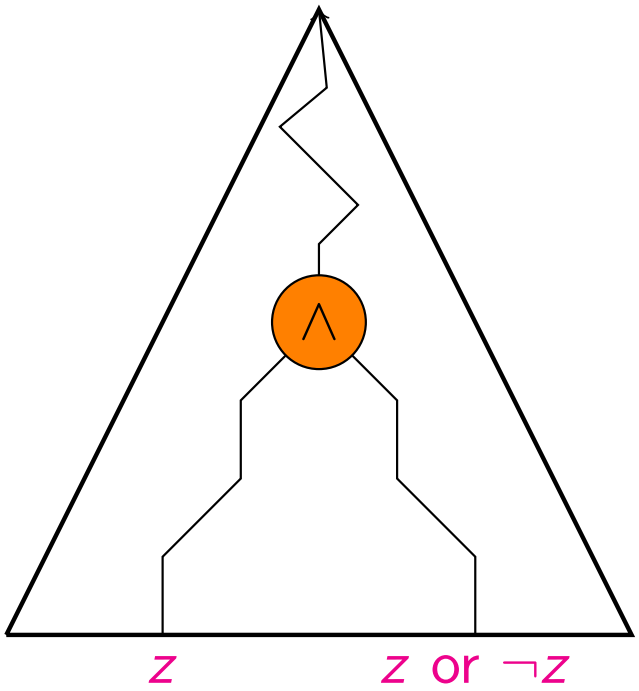

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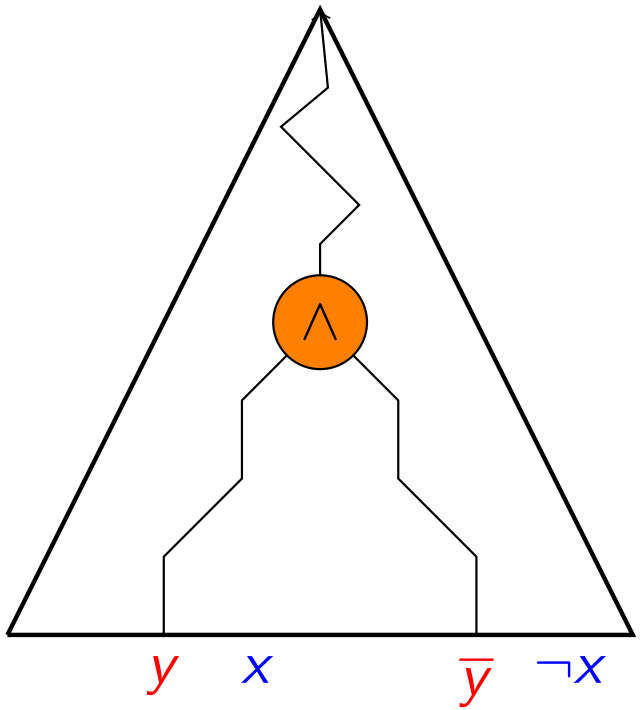

Disallowed paths

A specification in DNNF or wDNNF is already in SynNNF for any output sequence.

SynNNF is relatively succinct

There exist polynomial sized SynNNF specifications that only admit

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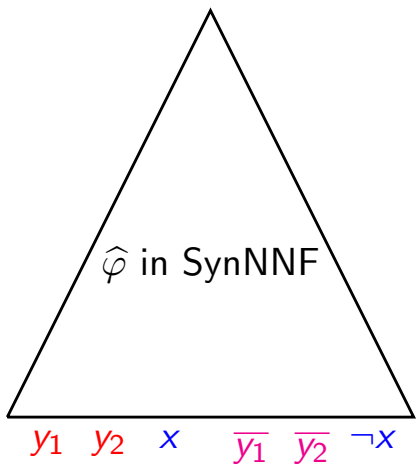
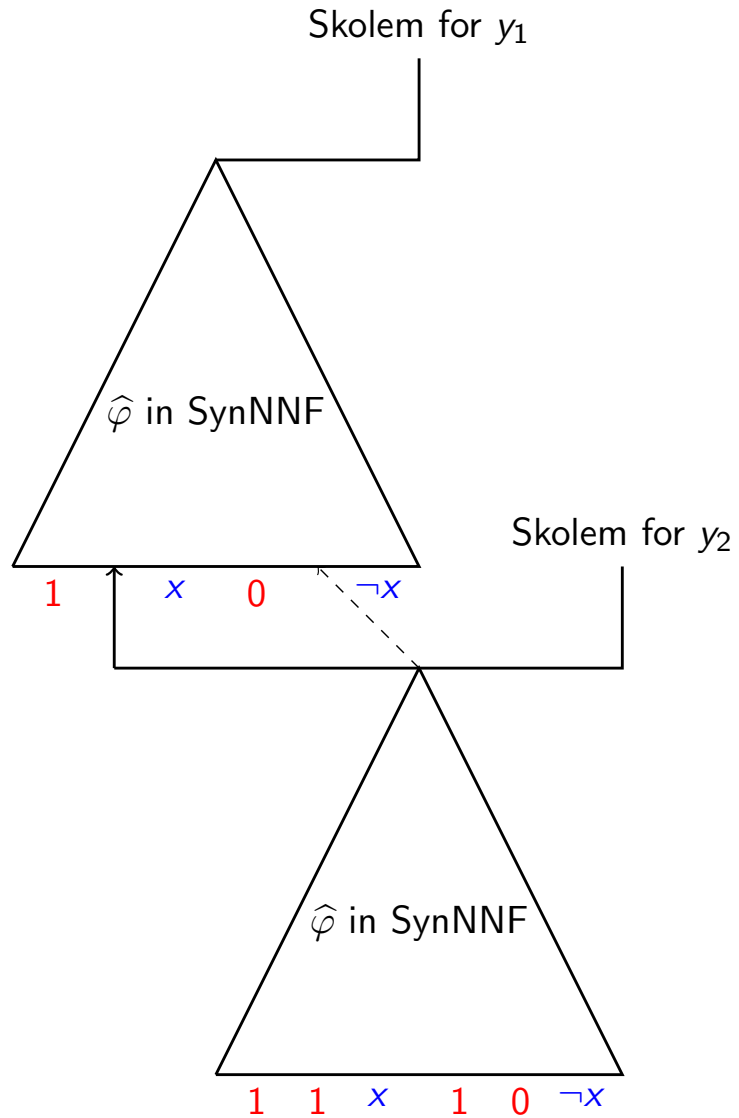
$$\text{NNF} \sqsubset \text{SynNNF} \sqsubset \text{wDNNF} \sqsubset \text{DNNF} \sqsubset \text{dDNNF} \sqsubseteq \text{BDD}$$

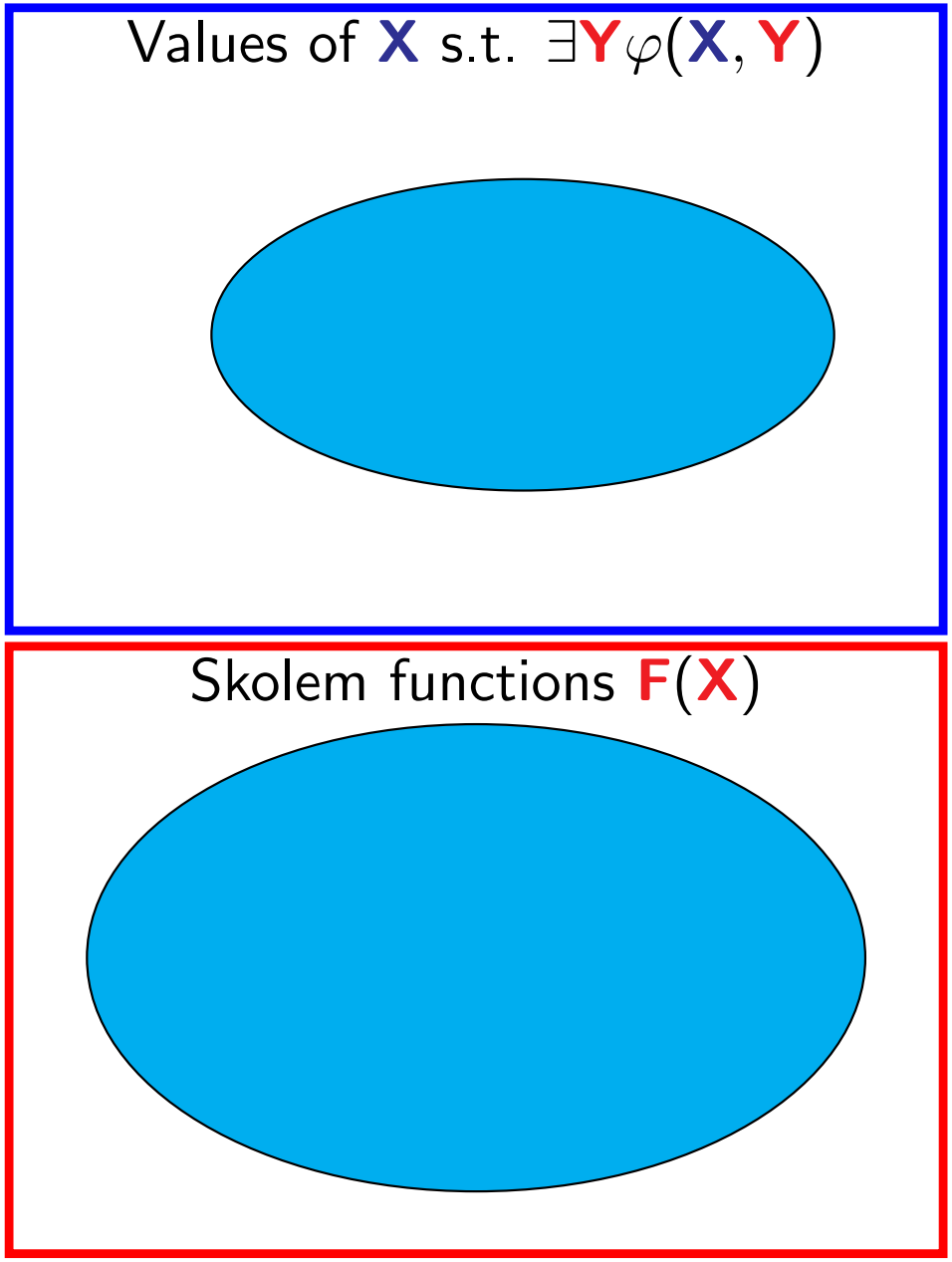
How does SynNNF help Skolem function synthesis?





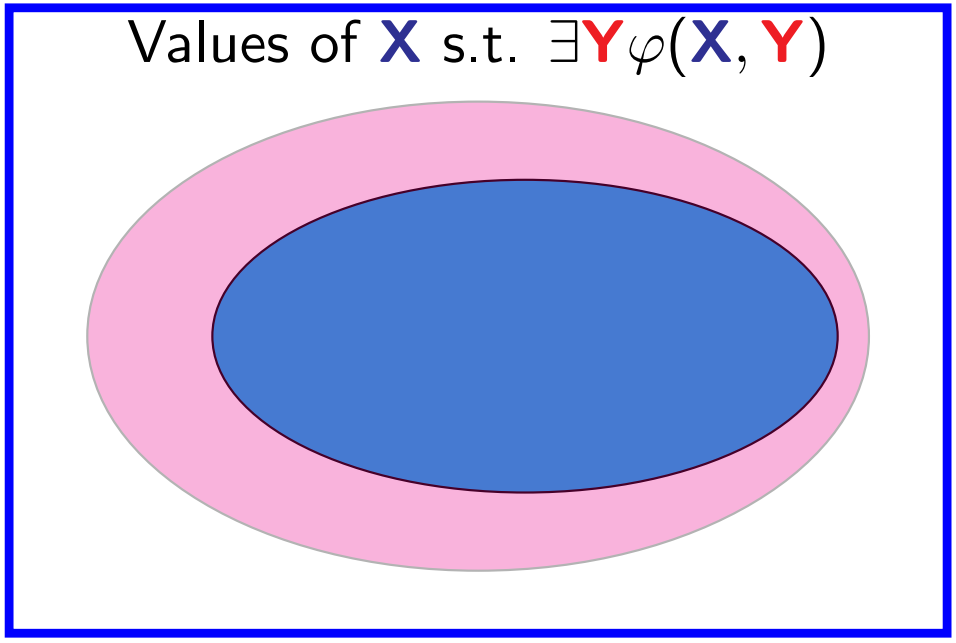
Synthesis: $m \times |\varphi|$ circuit size, $\mathcal{O}(m^2)$ additional wires.

Refinement w.r.t. synthesis



Given spec: $\varphi(\mathbf{X}, \mathbf{Y})$

Refinement w.r.t. synthesis



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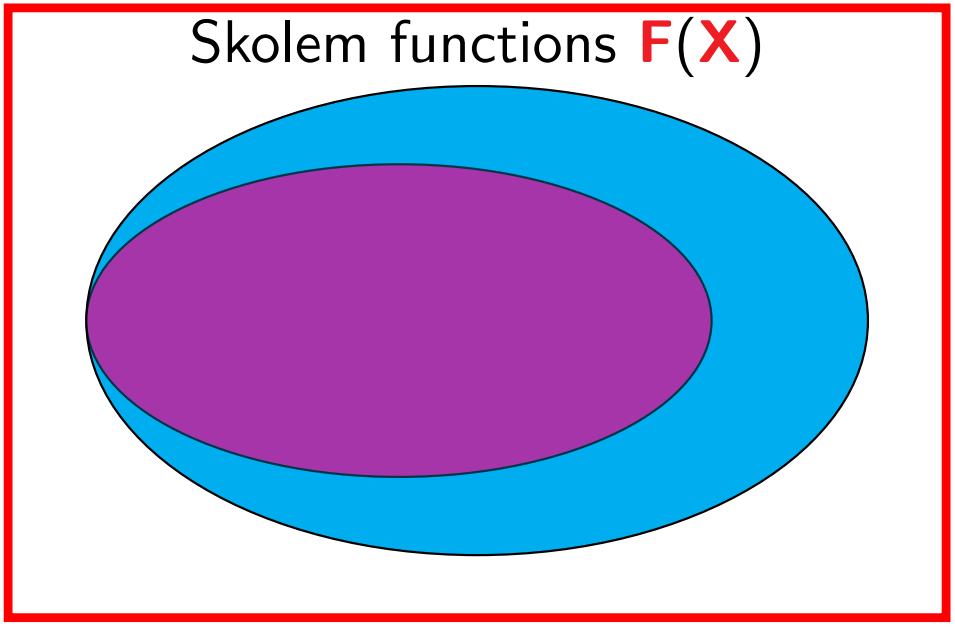
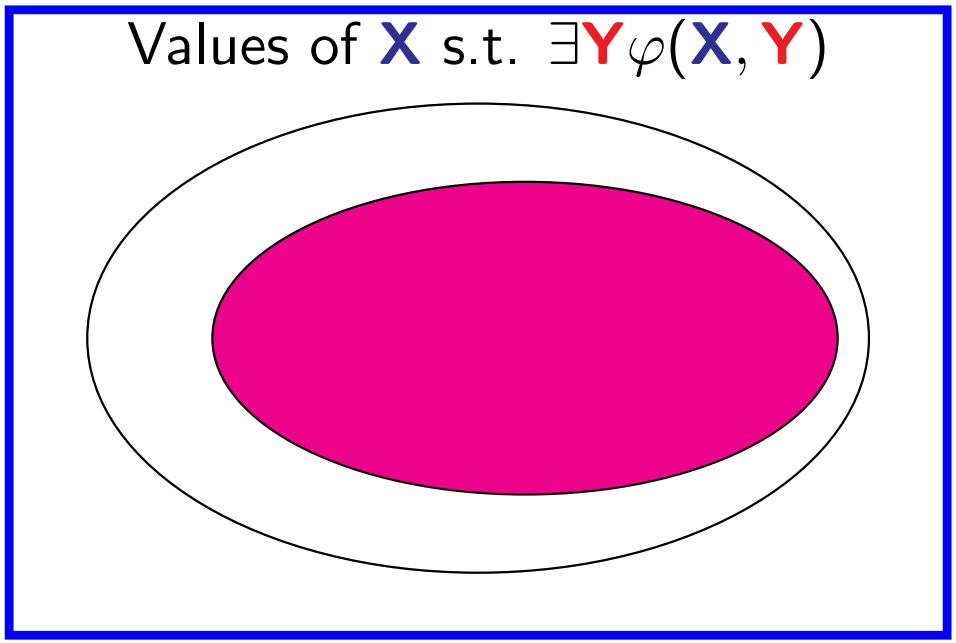


Refined spec: $\widetilde{\varphi}(\mathbf{X}, \mathbf{Y})$

$\widetilde{\varphi} \preceq_{syn} \varphi$

Refinement w.r.t. synthesis



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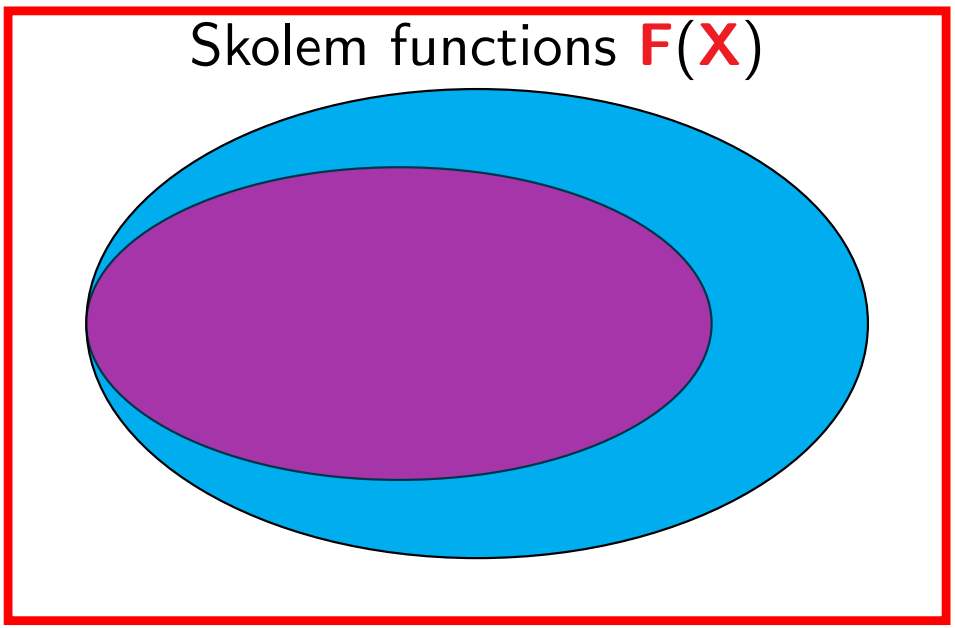

Strongly
Refined spec: $\widetilde{\varphi}(\mathbf{X}, \mathbf{Y})$
$\widetilde{\varphi} \preceq^{*}_{\mathit{syn}} \varphi$

Refinement w.r.t. synthesis

Lemma

If $\widetilde{F} \preceq_{syn} F$, every Skolem function vector for $\mathbf{Y}$ in $\widetilde{F}$ is also a Skolem function vector for $\mathbf{Y}$ in F.

Refinement w.r.t. synthesis

Lemma

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Example: $(y_2 \wedge y_1) \preceq_{syn}$
$((\neg y_1 \vee y_2 \vee x_1) \wedge (y_1 \vee \neg y_2) \wedge (y_1 \vee \neg x_1) \wedge (y_2 \vee x_2))$

Putting it all together

Tool C2Syn:

- Input: φ in CNF (or AIG)
- Output: Refined $\widetilde{\varphi}$ in SynNNF

- Branches only on output variables
- Aggressively tries to refine whenever possible
- Details in our FMCAD 2019 paper

Experimental Results

Comparison of run-time with

- CADET [Rabe et al 2016]
- BFSS [Akshay et al 2018]
- BDD [BDD pipeline of BFSS]

Benchmarks: QBFEVAL 2018 and Factorization (408 total)

Benchmarks (Total)	Compiled By C2Syn			BDD compilation	Total in SynNNF
	Stage I	Stage II	Total		
QBFEVAL (402)	103	83	186	153	**283**
FA.QD (6)	0	6	6	6	**6**

Table: Compilation into SynNNF

Experimental Results

Bench mark	C2Syn vs CADET		C2Syn vs BFSS		C2Syn \ (CADET ∪ BFSS)
	C2Syn\ CADET	CADET\ C2Syn	C2Syn\ BFSS	BFSS\ C2Syn	
QBFEVAL	78	105	83	78	**75**
FA.QD	2	0	3	0	**2**

Table: Comparison Results of C2Syn

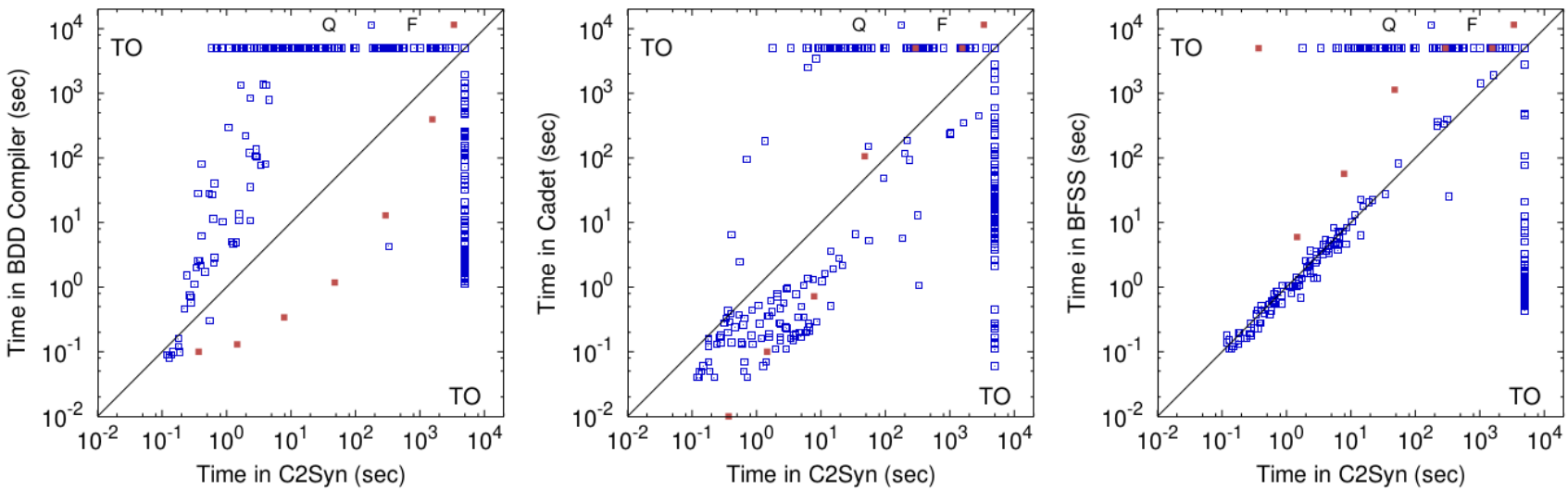

Fig. 1: Time comparisons: C2Syn vs BDD^{BFSS}, CADET, BFSS