Guessing Game

I’m thinking of a research area where:

• Algorithms have recently improved by *orders of magnitude*

• Computers solve tasks *better than* humans

• Computers solve tasks *without help* from humans

• *Big investments* are being made by the government and industry, including companies like:
  • Amazon, Apple, Facebook, Google, Intel, Microsoft

• It can be described using two letters; first one is A
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Philosophers have long dreamed of machines that can reason. The pursuit of this dream has occupied some of the best minds and led both to great achievements and great disappointments.
~1700
Leibniz – mechanized human reasoning

1928
Hilbert
Entscheidungsproblem

Church – lamda calculus
Turing – reduction halting problem

1936

1954
Davis – decision procedure for Presburger arithmetic
Automated Reasoning: A Failure?

• At the turn of the century, automated reasoning was still considered by many to be impractical for most real-world applications

• Interesting problems appeared to be beyond the reach of automated methods because of decidability and complexity barriers

• The dream of Hilbert's mechanized mathematics or Leibniz's calculating machine was believed by many to be simply unattainable
Princeton, c. 2000

- **Chaff SAT solver**: orders of magnitude faster than previous SAT solvers
- **Important observation**: many real-world problems do not exhibit worst-case theoretical performance

Palo Alto, c. 2001

- **Idea**: combine fast new SAT solvers with decision procedures for decidable first-order theories
- **SVC, CVC** solvers (Stanford); **ICS, Yices** solvers (SRI)
- **Satisfiability Modulo Theories** (SMT) was born
SMT solvers: *general-purpose* logic engines

- Given condition $X$, is it possible for $Y$ to happen
- $X$ and $Y$ are expressed in a *rich logical language*
  - First-order logic
  - Domain-specific reasoning
    - arithmetic, arrays, bit-vectors, data types, etc.

SMT solvers are *changing the way people solve problems*

- Instead of building a *special-purpose* solver
- *Translate* into a logical formula and use an SMT solver
- Not only easier, *often better*
SMT solvers are changing the way people solve problems

- Instead of building a special-purpose solver
- Translate into a logical formula and use an SMT solver
- Not only easier, often better
Evolution of SMT solving

- Total time on QF_BV benchmarks (virtual best)
  
  - Average speedup: 11X
  - Unsolved (2010): 3100
    - All but 200 solved now
    - Over 2000 now solved in less than 1 second
Zelkova

Security Policy

(allow, principal : *, action : getObject, resource : cs240/*, condition : (StringEquals, aws:sourceVpc, vpc-111bbb222), (StringLike, s3:prefix, cs240/Exam*))

SMT Encoding

a = “getObject” ∨ r = “cs240/*” ∧ vpcExists ∧ vpc = “vpc-111bbb222” ∧ s3PrefixExists ∧ “cs240/Exam” prefixOf s3Prefix

SMT Solvers (cvc5 and z3)

Strings and RegExp

Bitvectors

Arithmetic
Satisfiability Modulo Theories

Clark Barrett, Stanford University
SAT + SMT Winter School, Dec 15, 2023
**Acknowledgments**: Many thanks to Cesare Tinelli and Albert Oliveras for contributing some of the material used in these slides.

**Disclaimer**: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.
Introduction
SMT Solvers

- Arithmetic
- Arrays
- UF
- Bit-Vectors

Core

assertions

explanations
conflicts
lemmas
propagations

SAT Solver

DPLL
SAT Solver

- Only sees *Boolean skeleton* of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as *assertions*
SMT Solvers

Core
- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

Diagrams:
- Arithmetic
- Arrays
- UF
- Core
- SAT Solver
  - DPLL
- Assertions
  - explanation
  - conflicts
  - lemmas
  - propagation
SMT Solvers

Theory Solvers

- Decide $T$-satisfiability of a conjunction of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation
Theory Solvers
Given a theory $T$, a *Theory Solver* for $T$ takes as input a set $\Phi$ of literals and determines whether $\Phi$ is $T$-satisfiable.

$\Phi$ is $T$-satisfiable iff there is some model $M$ of $T$ such that each formula in $\Phi$ holds in $M$. 
Theories of Interest: UF

Equality ( = ) with Uninterpreted Functions [NO80, BD94, NO07]

Typically used to abstract unsupported constructs, e.g.,

- non-linear multiplication in arithmetic
- ALUs in circuits

Example: The formula

\[ a \ast (|b| + c) = d \land b \ast (|a| + c) \neq d \land a = b \]

is unsatisfiable, but no arithmetic reasoning is needed if we abstract it to

\[ \text{mul}(a, \text{add}(\text{abs}(b), c)) = d \land \text{mul}(b, \text{add}(\text{abs}(a), c)) \neq d \land a = b \]

it is still unsatisfiable
Theories of Interest: Arithmetic

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- **Bounds**: $x \, \preceq \, k$ with $\preceq \in \{<, >, \leq, \geq, =\}$ [BBC+05a]

- **Difference logic**: $x - y \, \preceq \, k$, with
  $\preceq \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, CM06]

- **UTVPI**: $\pm x \pm y \, \preceq \, k$, with $\preceq \in \{<, >, \leq, \geq, =\}$ [LM05]

- **Linear arithmetic**, e.g., $2x - 3y + 4z \leq 5$ [DdM06]

- **Non-linear arithmetic**, e.g.,
  $2xy + 4xz^2 - 5y \leq 10$ [BLNM+09, ZM10, JdM12]
Theories of Interest: Arrays

Used in software verification and hardware verification (for memories) [SBDL01, BNO+08a, dMB09]

Two interpreted function symbols read and write

Axiomatized by:

- $\forall a \forall i \forall v \text{read}(\text{write}(a, i, v), i) = v$
- $\forall a \forall i \forall j \forall v \ i \neq j \rightarrow \text{read}(\text{write}(a, i, v), j) = \text{read}(a, j)$

Sometimes also with extensionality:

- $\forall a \forall b \ (\forall i \text{read}(a, i) = \text{read}(b, i) \rightarrow a = b)$

Is the following set of literals satisfiable in this theory?

$\text{write}(a, i, x) \neq b$, $\text{read}(b, i) = y$, $\text{read}(\text{write}(b, i, x), j) = y$, $a = b$, $i = j$
Theories of Interest: Bit vectors

Useful both in hardware and software verification [BCF+07, BB09, HBJ+14]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- **String-like**: concat, extract, . . .
- **Logical**: bit-wise not, or, and, . . .
- **Arithmetic**: add, subtract, multiply, . . .
- **Comparison**: <, >, . . .

Is this formula satisfiable over bit vectors of size 3?

\[ a[1 : 0] \neq b[1 : 0] \land (a \mid b) = c \land c[0] = 0 \land a[1] + b[1] = 0 \]
We consider a simple example: difference logic.

In difference logic, we are interested in the satisfiability of a conjunction of arithmetic atoms.

Each atom is of the form $x - y \bowtie c$, where $x$ and $y$ are variables, $c$ is a numeric constant, and $\bowtie \in \{=, <, \leq, >, \geq\}$.

The variables can range over either the integers (QF_IDL) or the reals (QF_RDL).
The first step is to rewrite everything in terms of $\leq$:
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- $x - y = c \implies x - y \leq c \land x - y \geq c$
Difference Logic

The first step is to rewrite everything in terms of $\leq$:

- $x - y = c \implies x - y \leq c \land x - y \geq c$
- $x - y \geq c \implies y - x \leq -c$
Difference Logic

The first step is to rewrite everything in terms of $\leq$:

- $x - y = c$  $\implies$  $x - y \leq c$  $\land$  $x - y \geq c$
- $x - y \geq c$  $\implies$  $y - x \leq -c$
- $x - y > c$  $\implies$  $y - x < -c$
The first step is to rewrite everything in terms of $\leq$:

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- $x - y > c \implies y - x < -c$
- $x - y < c \implies x - y \leq c - 1$ (integers)
The first step is to rewrite everything in terms of $\leq$:

- $x - y = c \implies x - y \leq c \land x - y \geq c$
- $x - y \geq c \implies y - x \leq -c$
- $x - y > c \implies y - x < -c$
- $x - y < c \implies x - y \leq c - 1$ (integers)
- $x - y < c \implies x - y \leq c - \delta$ (reals)
Now we have a conjunction of literals, all of the form \( x - y \leq c \).

From these literals, we form a weighted directed graph with a vertex for each variable.

For each literal \( x - y \leq c \), there is an edge \( x \xrightarrow{c} y \).

The set of literals is satisfiable iff there is no cycle for which the sum of the weights on the edges is negative.

There are a number of efficient algorithms for detecting negative cycles in graphs.
\( x - y = 5 \land z - y \geq 2 \land z - x > 2 \land w - x = 2 \land z - w < 0 \)
Difference Logic Example

\[ x - y = 5 \land z - y \geq 2 \land z - x > 2 \land w - x = 2 \land z - w < 0 \]

\[ x - y = 5 \]
\[ z - y \geq 2 \]
\[ z - x > 2 \]
\[ w - x = 2 \]
\[ z - w < 0 \]
Difference Logic Example

\[ x - y = 5 \land z - y \geq 2 \land z - x > 2 \land w - x = 2 \land z - w < 0 \]

\[
\begin{align*}
x - y &= 5 \\
z - y &\geq 2 \\
z - x &> 2 \implies \\
w - x &= 2 \\
z - w &< 0
\end{align*}
\]
Difference Logic Example

\[ x - y = 5 \land z - y \geq 2 \land z - x > 2 \land w - x = 2 \land z - w < 0 \]

\begin{align*}
  x - y &= 5 \\
  z - y &\geq 2 \\
  z - x &> 2 \\
  w - x &= 2 \\
  z - w &< 0
\end{align*}

\begin{align*}
  x - y &= 5 \\
  x - y &\leq 5 \land y - x \leq -5 \\
  z - y &\geq 2 \\
  y - z &\leq -2 \\
  z - x &> 2 \Rightarrow x - z \leq -3 \\
  w - x &= 2 \\
  w - x &\leq 2 \land x - w \leq -2 \\
  z - w &< 0 \\
  z - w &\leq -1
\end{align*}
Difference Logic Example

24
DPLL($T$): Combining $T$-Solvers with SAT
**Def.** A formula is *(un)satisfiable in* a theory $T$, or $T$-(un)satisfiable, if there is a (no) model of $T$ that satisfies it.

**Note:** The $T$-satisfiability of quantifier-free formulas is decidable iff the $T$-satisfiability of conjunctions/sets of literals is decidable.

(Convert the formula in DNF and check if any of its disjuncts is $T$-sat)

**Problem:** In practice, dealing with Boolean combinations of literals is as hard as in propositional logic.

**Solution:** Exploit propositional satisfiability technology.
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Two main approaches:

1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]
   - translate into an equisatisfiable propositional formula
   - feed it to any SAT solver

   Notable systems: **UCLID**

2. "Lazy" [ACG00, dMR02, BDS02, ABC+02]
   - abstract the input formula to a propositional one
   - feed it to a (DPLL-based) SAT solver
   - use a theory decision procedure to refine the formula and guide the SAT solver

   Notable systems: *cvc5, MathSAT, OpenSMT, SMTInterpol, Yices, Z3*

This talk will focus on the lazy approach
Lifting SAT Technology to SMT

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(Very) Lazy Approach for SMT – Example

\[ g(a) = c \land f(g(a)) \neq f(c) \lor g(a) = d \land c \neq d \]

Theory \( T \): Equality with Uninterpreted Functions

Simplest setting:

- Off-line SAT solver
- Non-incremental theory solver for conjunctions of equalities and disequalities
- Theory atoms (e.g., \( g(a) = c \)) abstracted to propositional atoms (e.g., 1)
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(Very) Lazy Approach for SMT – Example

\[
g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d
\]

1

2

3

4

• Send \{1, 2 \lor 3, 4\} to SAT solver.

• SAT solver returns model \{1, 2, 4\}.

• Theory solver finds (concretization of) \{1, 2, 4\} unsat.

• Send \{1, 2 \lor 3, 4, 1 \lor 2 \lor 4\} to SAT solver.

• SAT solver returns model \{1, 3, 4\}.

• Theory solver finds \{1, 3, 4\} unsat.

• Send \{1, 2 \lor 3, 4, 1 \lor 2, 1 \lor 3 \lor 4\} to SAT solver.

• SAT solver finds \{1, 2 \lor 3, 4, 1 \lor 2, 1 \lor 3 \lor 4, 1 \lor 3 \lor 4\} unsat.

Done: the original formula is unsatisfiable in UF.
(Very) Lazy Approach for SMT – Example

\[
\begin{align*}
g(a) &= c & \land & & f(g(a)) \neq f(c) & \lor & & g(a) = d & \land & & c \neq d \\
\text{(1)} & & \land & & \text{(2)} & & \lor & & \text{(3)} & & \land & & \text{(4)}
\end{align*}
\]

- Send \(\{1, \bar{2} \lor 3, \bar{4}\}\) to SAT solver.
- SAT solver returns model \(\{1, \bar{2}, \bar{4}\}\).
  Theory solver finds (concretization of) \(\{1, \bar{2}, \bar{4}\}\) unsat.
- Send \(\{1, \bar{2} \lor 3, \bar{4}, \bar{1} \lor 2 \lor 4\}\) to SAT solver.
  SAT solver returns model \(\{1, 3, \bar{4}\}\).
  Theory solver finds \(\{1, 3, \bar{4}\}\) unsat.
- Send \(\{1, \bar{2} \lor 3, \bar{4}, \bar{1} \lor 2, \bar{1} \lor 3 \lor 4\}\) to SAT solver.
  SAT solver finds \(\{1, \bar{2} \lor 3, \bar{4}, \bar{1} \lor 2 \lor 4, \bar{1} \lor 3 \lor 4\}\) unsat.

Done: the original formula is unsatisfiable in UF.
(Very) Lazy Approach for SMT – Example

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g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d
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• SAT solver returns model \(\{1, \overline{2}, \overline{4}\}\).

  Theory solver finds (concretization of) \(\{1, \overline{2}, \overline{4}\}\) unsat.

• Send \(\{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4\}\) to SAT solver.

• SAT solver returns model \(\{1, 3, \overline{4}\}\).

  Theory solver finds \(\{1, 3, \overline{4}\}\) unsat.

• Send \(\{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2, \overline{1} \lor 3 \lor 4\}\) to SAT solver.

• SAT solver finds \(\{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4, \overline{1} \lor 3 \lor 4\}\) unsat.

Done: the original formula is unsatisfiable in UF.
(Very) Lazy Approach for SMT – Example

\[
g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d
\]

1

• Send \{1, \overline{2} \lor 3, \overline{4}\} to SAT solver.
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• Send \{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4\} to SAT solver.
  • SAT solver returns model \{1, 3, \overline{4}\}.
  Theory solver finds \{1, 3, \overline{4}\} unsat.
  • Send \{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor \overline{3} \lor 4\} to SAT solver.
  • SAT solver finds \{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4, \overline{1} \lor \overline{3} \lor 4\} unsat.
  Done: the original formula is unsatisfiable in UF.
(Very) Lazy Approach for SMT – Example

\[ g(a) = c \land (f(g(a)) \neq f(c)) \lor g(a) = d \land c \neq d \]

1. Send \{1, 2 \lor 3, 4\} to SAT solver.
2. SAT solver returns model \{1, 2, 4\}.
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3. Send \{1, 2 \lor 3, 4, \overline{1} \lor 2 \lor 4\} to SAT solver.
4. SAT solver returns model \{1, 3, 4\}.
   Theory solver finds \{1, 3, 4\} unsat.
5. Send \{1, 2 \lor 3, 4, \overline{1} \lor 2, \overline{1} \lor 3 \lor 4\} to SAT solver.
6. SAT solver finds \{1, 2 \lor 3, 4, \overline{1} \lor 2 \lor 4, \overline{1} \lor 3 \lor 4\} unsat.

Done: the original formula is unsatisfiable in UF.
(Very) Lazy Approach for SMT – Example

\[ g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d \]

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Done: the original formula is unsatisfiable in UF.
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\[ g(a) = c \land f(g(a)) \neq f(c) \lor g(a) = d \land c \neq d \]

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6. SAT solver finds \( \{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4, \overline{1} \lor \overline{3} \lor 4\} \) unsat.

\textit{Done: the original formula is unsatisfiable in UF.}
(Very) Lazy Approach for SMT – Example

\[ g(a) = c \land \begin{array}{l} f(g(a)) \neq f(c) \lor \begin{array}{l} g(a) = d \land c \neq d \end{array} \\
1 \end{array} \]

\[ \begin{array}{l} \begin{array}{l} \begin{array}{l} 1 \end{array} \end{array} \\
2 \end{array} \]

\[ \begin{array}{l} \begin{array}{l} \begin{array}{l} 3 \end{array} \end{array} \\
4 \end{array} \]

• Send \( \{1, \overline{2} \lor 3, \overline{4}\} \) to SAT solver.
• SAT solver returns model \( \{1, \overline{2}, \overline{4}\} \).
  Theory solver finds (concretization of) \( \{1, \overline{2}, \overline{4}\} \) unsat.
• Send \( \{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4\} \) to SAT solver.
• SAT solver returns model \( \{1, 3, \overline{4}\} \).
  Theory solver finds \( \{1, 3, \overline{4}\} \) unsat.
• Send \( \{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2, \overline{1} \lor \overline{3} \lor 4\} \) to SAT solver.
• SAT solver finds \( \{1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4, \overline{1} \lor \overline{3} \lor 4\} \) unsat.

*Done: the original formula is unsatisfiable in UF.*
Several **enhancements** are possible to **increase efficiency**: 

- Check $T$-satisfiability only of full propositional model
- Check $T$-satisfiability of partial assignment $M$ as it grows
- If $M$ is $T$-unsatisfiable, identify a $T$-unsatisfiable subset $M_0$ of $M$ and add $\neg M_0$ as a clause
- If $M$ is $T$-unsatisfiable, backtrack to some point where the assignment was still $T$-satisfiable
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Lazy Approach – Enhancements

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- If $M$ is $T$-unsatisfiable, add clause and restart
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Lazy Approach – Main Benefits

- Every tool does what it is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
  - SAT and theory solvers communicate via a simple API [GHN+04]
  - SMT for a new theory only requires new theory solver
  - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)
Lazy Approach – Main Benefits

• Every tool *does* what it is *good at*:
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Several variants and enhancements of lazy SMT solvers exist.

They can be modeled abstractly and declaratively as transition systems.

A transition system is a binary relation over states, induced by a set of conditional transition rules.

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07].
Advantages of Abstract Framework

An abstract framework helps one:

- skip over implementation details and unimportant control aspects
- reason formally about solvers for SAT and SMT
- model advanced features such as non-chronological backtracking, lemma learning, theory propagation, …
- describe different strategies and prove their correctness
- compare different systems at a higher level
- get new insights for further enhancements

The one described next is a re-elaboration of those in [NOT06, KG07]
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The one described next is a re-elaboration of those in [NOT06, KG07]
The Original DPLL Procedure

- Modern SAT solvers are based on the **DPLL** procedure [DP60, DLL62]

- DPLL tries to **build** incrementally a **satisfying truth assignment** $M$ for a CNF formula $F$

- $M$ is grown by
  - **deducing** the truth value of a literal from $M$ and $F$, or
  - **guessing** a truth value

- If a wrong guess for a literal leads to an inconsistency, the procedure **backtracks** and tries the opposite value
States:

\text{fail or } \langle M, F \rangle

where

- \textit{M} is a sequence of literals and \textit{decision points} \bullet denoting a partial truth \textit{assignment}
- \textit{F} is a set of clauses denoting a CNF \textit{formula}

**Def.** If \( M = M_0 \bullet M_1 \bullet \cdots \bullet M_n \) where each \( M_i \) contains no decision points

- \( M_i \) is \textit{decision level} \( i \) of \( M \)
- \( M[i] \) \textit{def} \( M_0 \bullet \cdots \bullet M_i \)
An Abstract Framework for DPLL

States:

\[ \text{fail} \quad \text{or} \quad \langle M, F \rangle \]

Initial state:

- \( \langle (), F_0 \rangle \), where \( F_0 \) is to be checked for satisfiability

Expected final states:

- \( \text{fail} \) if \( F_0 \) is unsatisfiable
- \( \langle M, G \rangle \) otherwise, where
  - \( G \) is equivalent to \( F_0 \) and
  - \( M \) satisfies \( G \)
States treated like records:

- $M$ denotes the truth assignment component of current state
- $F$ denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

\[
\frac{p_1 \quad \cdots \quad p_n}{[M := e_1] \quad [F := e_2]}
\]

updating $M, F$ or both when premises $p_1, \ldots, p_n$ all hold
Transition Rules for the Original DPLL

Extending the assignment

**Propagate** \[ l_1 \lor \cdots \lor l_n \lor l \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M \quad l, \bar{l} \notin M \]

\[ M := M \cdot l \]

**Note:** When convenient, treat \( M \) as a set

**Note:** Clauses are treated modulo ACI of \( \lor \)

**Decide** \[ l \in \text{Lit}(F) \quad l, \bar{l} \notin M \]

\[ M := M \cdot l \]

**Note:** \( \text{Lit}(F) \) \( \overset{\text{def}}{=} \{ l \mid l \text{ literal of } F \} \cup \{ \bar{l} \mid l \text{ literal of } F \} \)
Transition Rules for the Original DPLL

Extending the assignment

**Propagate**

\[
\frac{l_1 \lor \cdots \lor l_n \lor l \in F}{M := M \cdot l}
\]

\[
\bar{l}_1, \ldots, \bar{l}_n \in M \quad l, \bar{l} \notin M
\]

**Note:** When convenient, treat \( M \) as a set

**Note:** Clauses are treated modulo ACI of \( \lor \)

**Decide**

\[
\frac{l \in \text{Lit}(F) \quad l, \bar{l} \notin M}{M := M \cdot l}
\]

**Note:** \( \text{Lit}(F) \stackrel{\text{def}}{=} \{l \mid \text{l literal of } F\} \cup \{\bar{l} \mid \text{l literal of } F\} \)
Transition Rules for the Original DPLL

Repairing the assignment

\[
\text{Fail} \quad \frac{l_1 \lor \cdots \lor l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M \quad \bullet \notin M}{\text{fail}}
\]

Backtrack

\[
l_1 \lor \cdots \lor l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M \quad M = M \bullet \bar{N} \quad \bullet \notin N
\]

\[
M := M \bar{l}
\]

Note: Last premise of Backtrack enforces chronological backtracking
Transition Rules for the Original DPLL

Repairing the assignment

\[
\text{Fail} \quad \frac{l_1 \lor \cdots \lor l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M \quad \bullet \notin M}{\text{fail}}
\]

\[
\text{Backtrack} \quad \frac{l_1 \lor \cdots \lor l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M \quad M = M \bullet l N \quad \bullet \notin N}{M := M \bar{l}}
\]

\textbf{Note:} Last premise of \textbf{Backtrack} enforces \textit{chronological} backtracking
To model conflict-driven backjumping and learning, add to states a third component $C$ whose value is either no or a conflict clause.

States: fail or $\langle M, F, C \rangle$

Initial state:

- $\langle (), F_0, \text{no} \rangle$, where $F_0$ is to be checked for satisfiability

Expected final states:

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  - $G$ is equivalent to $F_0$ and
  - $M$ satisfies $G$
Replace **Backtrack** with

**Conflict**
\[
C = \text{no} \quad l_1 \lor \cdots \lor l_n \in F \quad \overline{l}_1, \ldots, \overline{l}_n \in M
\]
\[
C := l_1 \lor \cdots \lor l_n
\]

**Explain**
\[
C = l \lor D \quad l_1 \lor \cdots \lor l_n \lor \overline{l} \in F \quad \overline{l}_1, \ldots, \overline{l}_n <_M \overline{l}
\]
\[
C := l_1 \lor \cdots \lor l_n \lor D
\]

**Backjump**
\[
C = l_1 \lor \cdots \lor l_n \lor l \quad \text{lev} \ \overline{l}_1, \ldots, \text{lev} \ \overline{l}_n \leq i < \text{lev} \ \overline{l}
\]
\[
C := \text{no} \quad M := M^{[i]} \ l
\]

Maintain invariant: \( F \models_p C \) and \( M \models_p \neg C \) when \( C \neq \text{no} \)

**Note:** \( \models_p \) denotes propositional entailment
From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

**Conflict**

\[
C = \text{no} \quad l_1 \lor \cdots \lor l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M
\]

\[
C := l_1 \lor \cdots \lor l_n
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**Explain**

\[
C = l \lor D \quad l_1 \lor \cdots \lor l_n \lor \bar{l} \in F \quad \bar{l}_1, \ldots, \bar{l}_n <_M \bar{l}
\]

\[
C := l_1 \lor \cdots \lor l_n \lor D
\]

**Backjump**

\[
C = l_1 \lor \cdots \lor l_n \lor l \quad \text{lev} \bar{l}_1, \ldots, \text{lev} \bar{l}_n \leq i < \text{lev} \bar{l}
\]

\[
C := \text{no} \quad M := M[i] \, l
\]

**Note:** \( l <_M l' \) if \( l \) occurs before \( l' \) in \( M \)

\( \text{lev} \, l = i \) iff \( l \) occurs in decision level \( i \) of \( M \)

Maintain invariant: \( F \models_p C \) and \( M \models_p \neg C \) when \( C \neq \text{no} \)

**Note:** \( \models_p \) denotes propositional entailment
Replace **Backtrack** with

**Conflict**

\[
C = \text{no} \quad l_1 \lor \cdots \lor l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M \\
C := l_1 \lor \cdots \lor l_n
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**Explain**

\[
C = l \lor D \quad l_1 \lor \cdots \lor l_n \lor \bar{l} \in F \quad \bar{l}_1, \ldots, \bar{l}_n <_M \bar{l} \\
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C = l_1 \lor \cdots \lor l_n \lor l \quad \text{lev } \bar{l}_1, \ldots, \text{lev } \bar{l}_n \leq i < \text{lev } \bar{l} \\
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Maintain **invariant**: \( F \models_p C \) and \( M \models_p \neg C \) when \( C \neq \text{no} \)

**Note**: \( \models_p \) denotes propositional entailment
Modify \textbf{Fail} to

\[
\text{Fail} \quad \frac{C \neq \text{no} \quad \bullet \notin M}{\text{fail}}
\]
Modify \textbf{Fail} to

\begin{align*}
\text{Fail} & \quad C \not\equiv \text{no} \quad \bullet \not\in M \\
& \quad \text{fail}
\end{align*}
**Execution Example**

$F := \{1, \overline{1} \lor 2, \overline{3} \lor 4, \overline{5} \lor 6, \overline{1} \lor \overline{5} \lor 7, \overline{2} \lor \overline{5} \lor 6 \lor 7\}$

<table>
<thead>
<tr>
<th>M</th>
<th>F</th>
<th>C</th>
<th>rule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F$</td>
<td>no</td>
<td>by Propagate</td>
</tr>
<tr>
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<td>$F$</td>
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<td>by Propagate</td>
</tr>
<tr>
<td>12</td>
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<td>by Decide</td>
</tr>
<tr>
<td></td>
<td>$F$</td>
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</tr>
<tr>
<td></td>
<td>$F$</td>
<td>no</td>
<td>by Conflict</td>
</tr>
<tr>
<td></td>
<td>$F$</td>
<td>$2 \lor 5 \lor 6 \lor 7$</td>
<td>by Explain with $\overline{1} \lor \overline{5} \lor 7$</td>
</tr>
<tr>
<td></td>
<td>$F$</td>
<td>$\overline{1} \lor \overline{2} \lor \overline{5} \lor 6$</td>
<td>by Explain with $\overline{5} \lor 6$</td>
</tr>
<tr>
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<td>$F$</td>
<td>no</td>
<td>by Backjump</td>
</tr>
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<td>by Decide</td>
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<td>$F$</td>
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<td>by Decide</td>
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<td>...</td>
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</tbody>
</table>
Execution Example

\[ F := \{1, \overline{1} \lor 2, \overline{3} \lor 4, \overline{5} \lor \overline{6}, \overline{1} \lor \overline{5} \lor 7, \overline{2} \lor \overline{5} \lor 6 \lor 7\} \]

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<tr>
<td>1 2</td>
<td>F</td>
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</tr>
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<td>1 2 3</td>
<td>F</td>
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</tr>
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<td>by Propagate</td>
</tr>
<tr>
<td>1 2 3 4 5 6 7</td>
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<td>no</td>
<td>by Propagate</td>
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<td></td>
<td>by Conflict</td>
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<tr>
<td>1 2 3 4 5 6 7 3</td>
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<td></td>
<td>by Explain with \overline{1} \lor \overline{5} \lor 7</td>
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<tr>
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<td></td>
<td>by Explain with \overline{5} \lor \overline{6}</td>
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<td>by Backjump</td>
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<td>...</td>
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</tr>
<tr>
<td>3</td>
<td>F</td>
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<td>by Decide</td>
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</table>

...
Execution Example

\[ F := \{1, \overline{1} \lor 2, \overline{3} \lor 4, \overline{5} \lor 6, \overline{1} \lor \overline{5} \lor 7, \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\} \]

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<td>by Propagate</td>
</tr>
<tr>
<td>1 2</td>
<td>( F )</td>
<td>no</td>
<td>by Propagate</td>
</tr>
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<td>( F )</td>
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<td>by Propagate</td>
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<tr>
<td>1 2 3 4 5 6 7</td>
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<td>by Propagate</td>
</tr>
<tr>
<td>1 2 3 4 5 6 7</td>
<td>2 \lor 5 \lor 6 \lor \overline{7}</td>
<td>( F )</td>
<td>by Conflict</td>
</tr>
<tr>
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<td>( F )</td>
<td>by Explain with ( \overline{1} \lor 5 \lor 7 )</td>
</tr>
<tr>
<td>1 2 3 4 5 6 7</td>
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Execution Example

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Execution Example

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### Execution Example

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<td>by Conflict</td>
</tr>
<tr>
<td>1 2 \cdot 3 4 \cdot 5 \bar{6} \bar{7} \bar{6} \bar{6} \bar{6}</td>
<td>(F)</td>
<td>(\bar{1} \lor 2 \lor \bar{5} \lor 6)</td>
<td>by Explain with (\bar{1} \lor \bar{5} \lor 7)</td>
</tr>
<tr>
<td>1 2 \cdot 3 4 \cdot 5 \bar{6} \bar{7} \bar{6} \bar{6} \bar{6} \bar{6}</td>
<td>(F)</td>
<td>(\bar{1} \lor 2 \lor \bar{5})</td>
<td>by Explain with (\bar{5} \lor 6)</td>
</tr>
<tr>
<td>1 2 \cdot 3 4 \cdot 5 \bar{6} \bar{7} \bar{6} \bar{6} \bar{6} \bar{6} \bar{6}</td>
<td>(F)</td>
<td>\text{no}</td>
<td>by Backjump</td>
</tr>
<tr>
<td>1 2 5 \cdot 3</td>
<td>(F)</td>
<td>\text{no}</td>
<td>by Decide</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Also add

**Learn**

\[
F \models_C C \quad C \notin F
\]

\[
F := F \cup \{C\}
\]

**Forget**

\[
C = \text{no} \quad F = G \cup \{C\} \quad G \models_C C
\]

\[
F := G
\]

**Restart**

\[
M := M^{[0]} \quad C := \text{no}
\]

**Note:** Learn can be applied to any clause stored in \(C\) when \(C \neq \text{no}\)
At the core, current CDCL SAT solvers are implementations of the transition system with rules

**Propagate, Decide,**

**Conflict, Explain, Backjump,**

**Learn, Forget, Restart**

\[
\text{Basic DPLL} \overset{\text{def}}{=} \{ \text{Propagate, Decide, Conflict, Explain, Backjump} \}
\]

\[
\text{DPLL} \overset{\text{def}}{=} \text{Basic DPLL} + \{ \text{Learn, Forget, Restart} \}
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The Basic DPLL System – Correctness

Some terminology:

**Irreducible state:** state for which no Basic DPLL rules apply

**Execution:** sequence of transitions allowed by the rules and starting with \( M = \emptyset \) and \( C = \text{no} \)

**Exhausted execution:** execution ending in an irreducible state

**Proposition** (Soundness) For every exhausted execution starting with \( F = F_0 \) and ending with fail, the clause set \( F_0 \) is unsatisfiable.

**Proposition** (Completeness) For every exhausted execution starting with \( F = F_0 \) and ending with \( C = \text{no} \), the clause set \( F_0 \) is satisfied by \( M \).
Some terminology:

*Irreducible state*: state for which no Basic DPLL rules apply

*Execution*: sequence of transitions allowed by the rules and starting with $M = \emptyset$ and $C = \text{no}$

*Exhausted execution*: execution ending in an irreducible state

**Proposition** (Strong Termination) Every execution in Basic DPLL is finite.

**Note**: This is not so immediate, because of **Backjump**.

**Proposition** (Soundness) For every exhausted execution starting with $F = F_0$ and ending with fail, the clause set $F_0$ is unsatisfiable.

**Proposition** (Completeness) For every exhausted execution starting
The Basic DPLL System – Correctness

Some terminology:

**Irreducible state**: state for which no Basic DPLL rules apply

**Execution**: sequence of transitions allowed by the rules and starting with $M = \emptyset$ and $C = no$

**Exhausted execution**: execution ending in an irreducible state

**Proposition** (Strong Termination) Every execution in Basic DPLL is finite.

**Lemma** Every exhausted execution ends with either $C = no$ or fail.

**Proposition** (Soundness) For every exhausted execution starting with $F = F_0$ and ending with fail, the clause set $F_0$ is unsatisfiable.

**Proposition** (Completeness) For every exhausted execution starting with $F = F_0$ and ending with $C = no$, the clause set $F_0$ is satisfied by $M$. 
Some terminology:

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The DPLL System – Strategies

• Applying
  • one Basic DPLL rule between each two **Learn** applications and
  • **Restart** less and less often

ensures termination

• A common basic strategy applies the rules with the following priorities:
  1. If $n > 0$ conflicts have been found so far, increase $n$ and apply **Restart**
  2. If a clause is falsified by $M$, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
  4. **Apply Learn**
  5. **Apply Backjump**
  6. Apply **Propagate** to completion
  7. **Apply Decide**
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From SAT to SMT

Same states and transitions but

- $F$ contains quantifier-free clauses in some theory $T$
- $M$ is a sequence of theory literals and decision points
- the DPLL system is augmented with rules $T$-Conflict, $T$-Propagate, $T$-Explain
- maintains invariant: $F \models_T C$ and $M \models_p \neg C$ when $C \neq \text{no}$

**Def.** $F \models_T G$ iff every model of $T$ that satisfies $F$ satisfies $G$ as well
SMT-level Rules

Fix a theory $T$

$T$-Conflict: $\frac{C = \text{no} \quad l_1, \ldots, l_n \in M \quad l_1, \ldots, l_n \models_T \bot}{C := \overline{l}_1 \lor \cdots \lor \overline{l}_n}$

$T$-Propagate: $\frac{l \in \text{Lit}(F) \quad M \models_T l \quad l, \overline{l} \notin M}{M := M \ l}$

$T$-Explain: $\frac{C = l \lor D \quad \overline{l}_1, \ldots, \overline{l}_n \models_T \overline{l} \quad \overline{l}_1, \ldots, \overline{l}_n \prec_M \overline{l}}{C := l_1 \lor \cdots \lor l_n \lor D}$

Note: $\bot = \text{empty clause}$

Note: $\models_T$ decided by theory solver
SMT-level Rules

Fix a theory \( T \)

**\( T \)-Conflict**

\[
\begin{align*}
C &= \text{no} \quad l_1, \ldots, l_n \in M \quad l_1, \ldots, l_n \models_T \bot \\
C &:= \overline{l}_1 \lor \cdots \lor \overline{l}_n
\end{align*}
\]

**\( T \)-Propagate**

\[
\begin{align*}
l &\in \text{Lit}(F) \quad M \models_T l \quad l, \overline{l} \not\in M \\
M &:= M \ u
\end{align*}
\]

**\( T \)-Explain**

\[
\begin{align*}
C &= l \lor D \quad \overline{l}_1, \ldots, \overline{l}_n \models_T \overline{l} \quad \overline{l}_1, \ldots, \overline{l}_n \not\models_M \overline{l} \\
C &:= l_1 \lor \cdots \lor l_n \lor D
\end{align*}
\]

**Note:** \( \bot \) = empty clause

**Note:** \( \models_T \) decided by theory solver
SMT-level Rules

Fix a theory $T$

**$T$-Conflict**

$\frac{C = \text{no } \ l_1, \ldots, l_n \in M \ l_1, \ldots, l_n \models_T \bot}{C := \overline{l}_1 \lor \cdots \lor \overline{l}_n}$

**$T$-Propagate**

$\frac{l \in \text{Lit}(F) \ M \models_T l \ l, \overline{l} \notin M}{M := M \ l}$

**$T$-Explain**

$\frac{C = l \lor D \ \overline{l}_1, \ldots, \overline{l}_n \models_T \overline{l} \ \overline{l}_1, \ldots, \overline{l}_n \prec_M \overline{l}}{C := l_1 \lor \cdots \lor l_n \lor D}$

**Note:** $\bot = \text{empty clause}$

**Note:** $\models_T$ decided by theory solver
Modeling the Very Lazy Theory Approach

**$T$-Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier UF example.

$$g(a) = c \land f(g(a)) \neq f(c) \lor g(a) = d \land c \neq d$$

<table>
<thead>
<tr>
<th>M</th>
<th>F</th>
<th>C</th>
<th>rule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>no</td>
<td>by Propagate $^+$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>no</td>
<td>by Decide</td>
</tr>
<tr>
<td>1 2</td>
<td>3</td>
<td>4</td>
<td>by $T$-Conflict</td>
</tr>
<tr>
<td>1 2</td>
<td>3</td>
<td>4</td>
<td>by Learn</td>
</tr>
<tr>
<td>1 2</td>
<td>3</td>
<td>4</td>
<td>by Restart</td>
</tr>
<tr>
<td>1 2</td>
<td>3</td>
<td>4</td>
<td>by Propagate $^+$</td>
</tr>
<tr>
<td>1 2</td>
<td>3</td>
<td>4</td>
<td>by $T$-Conflict, Learn</td>
</tr>
<tr>
<td>fail</td>
<td></td>
<td></td>
<td>by Fail</td>
</tr>
</tbody>
</table>
Modeling the Very Lazy Theory Approach

\[
\begin{align*}
g(a) &= c \
&\quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \
&\quad \land \quad c \neq d
\end{align*}
\]

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<tr>
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<td></td>
<td>no</td>
<td>by <strong>Propagate</strong>^+</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>no</td>
<td>by <strong>Decide</strong></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>no</td>
<td>by <strong>T-Conflict</strong></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>no</td>
<td>by <strong>Learn</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td>no</td>
<td>by <strong>Restart</strong></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>no</td>
<td>by <strong>Propagate</strong>^+</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>no</td>
<td>by <strong>T-Conflict, Learn</strong></td>
</tr>
<tr>
<td>fail</td>
<td>fail</td>
<td>fail</td>
<td>by <strong>Fail</strong></td>
</tr>
</tbody>
</table>
Modeling the Very Lazy Theory Approach

\[ g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d \]

<table>
<thead>
<tr>
<th>M</th>
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</thead>
<tbody>
<tr>
<td>1, ( \overline{2} \lor 3, 4 )</td>
<td>no</td>
<td>1, ( \overline{2} \lor 3, 4 )</td>
<td>no, by Propagate$^+$</td>
</tr>
<tr>
<td>1, ( \overline{4} )</td>
<td>no</td>
<td>1, ( \overline{2} \lor 3, 4 )</td>
<td>no, by Decide</td>
</tr>
<tr>
<td>1, ( \overline{4} \lor 2 )</td>
<td>no</td>
<td>1, ( \overline{2} \lor 3, 4 )</td>
<td>no, by T-Conflict</td>
</tr>
<tr>
<td>1, ( \overline{4} \lor 2 )</td>
<td>no</td>
<td>1, ( \overline{2} \lor 3, 4, \overline{1} \lor 2 \lor 4 )</td>
<td>no, by Learn</td>
</tr>
<tr>
<td>1, ( \overline{4} \lor 2 )</td>
<td>no</td>
<td>1, ( \overline{2} \lor 3, 4, \overline{1} \lor 2 \lor 4 )</td>
<td>no, by Restart</td>
</tr>
<tr>
<td>1, ( \overline{4} \lor 2 )</td>
<td>no</td>
<td>1, ( \overline{2} \lor 3, 4, \overline{1} \lor 2 \lor 4 )</td>
<td>no, by Propagate$^+$</td>
</tr>
<tr>
<td>1, ( \overline{4} \lor 2 )</td>
<td>no</td>
<td>1, ( \overline{2} \lor 3, 4, \overline{1} \lor 2 \lor 4 )</td>
<td>no, by T-Conflict, Learn</td>
</tr>
<tr>
<td>fail</td>
<td>fail</td>
<td>1, ( \overline{2} \lor 3, 4, \overline{1} \lor 2 \lor 4, \overline{1} \lor 3 \lor 4 )</td>
<td>no, by Fail</td>
</tr>
</tbody>
</table>
Modeling the Very Lazy Theory Approach

\[
g(a) = c \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{2} \quad \lor \quad \underbrace{g(a) = d \wedge c \neq d}_{4}
\]

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<tbody>
<tr>
<td>1 (\overline{4})</td>
<td>1, (\overline{2}\lor 3, \overline{4})</td>
<td>no</td>
<td>by Propagate$^+$</td>
</tr>
<tr>
<td>1 (\overline{4})</td>
<td>1, (\overline{2}\lor 3, \overline{4})</td>
<td>no</td>
<td>by Decide</td>
</tr>
<tr>
<td>1 (\overline{4}) (\cdot) 2</td>
<td>1, (\overline{2}\lor 3, \overline{4})</td>
<td>no</td>
<td>by T-Conflict</td>
</tr>
<tr>
<td>1 (\overline{4}) (\cdot) 2</td>
<td>1, (\overline{2}\lor 3, \overline{4})</td>
<td>(\overline{1}\lor 2\lor 4)</td>
<td>by Learn</td>
</tr>
<tr>
<td>1 (\overline{4}) (\cdot) 2</td>
<td>1, (\overline{2}\lor 3, \overline{4})</td>
<td>(\overline{1}\lor 2\lor 4)</td>
<td>by Restart</td>
</tr>
<tr>
<td>1 (\overline{4}) (\cdot) 2 (\cdot) 3</td>
<td>1, (\overline{2}\lor 3, \overline{4})</td>
<td>(\overline{1}\lor 2\lor 4)</td>
<td>by Propagate$^+$</td>
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<tr>
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<td>1, (\overline{2}\lor 3, \overline{4})</td>
<td>(\overline{1}\lor 2\lor 4)</td>
<td>by T-Conflict, Learn</td>
</tr>
<tr>
<td>1 (\overline{4}) (\cdot) 2 (\cdot) 3</td>
<td>fail</td>
<td>(\overline{1}\lor 3\lor 4)</td>
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<tbody>
<tr>
<td>1 4</td>
<td>1, 2 ∨ 3, 4</td>
<td>no</td>
<td>by Propagate⁺</td>
</tr>
<tr>
<td>1 4</td>
<td>1, 2 ∨ 3, 4</td>
<td>no</td>
<td>by Decide</td>
</tr>
<tr>
<td>1 4</td>
<td>1, 2 ∨ 3, 4</td>
<td>no</td>
<td>by T-Conflict</td>
</tr>
<tr>
<td>1 4</td>
<td>1, 2 ∨ 3, 4, 1 ∨ 2 ∨ 4</td>
<td>no</td>
<td>by Learn</td>
</tr>
<tr>
<td>1 4</td>
<td>1, 2 ∨ 3, 4, 1 ∨ 2 ∨ 4</td>
<td>no</td>
<td>by Restart</td>
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<tr>
<td>1 4 2 3</td>
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<td>no</td>
<td>by Propagate⁺</td>
</tr>
<tr>
<td>1 4 2 3</td>
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<td>fail</td>
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<td>1 4 2 3</td>
<td>1, 2 ∨ 3, 4, 1 ∨ 2 ∨ 4, 1 ∨ 3 ∨ 4</td>
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<td>no</td>
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<tr>
<td>1 #2</td>
<td>1, 2 \lor 3, 4</td>
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<tr>
<td>1 #2</td>
<td>1, 2 \lor 3, 4</td>
<td>\bar{1} \lor 2 \lor 4</td>
<td>by T-Conflict</td>
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<tr>
<td>1 #2</td>
<td>1, 2 \lor 3, 4, \bar{1} \lor 2 \lor 4</td>
<td>\bar{1} \lor 2 \lor 4</td>
<td>by Learn</td>
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<tr>
<td>1 4</td>
<td>1, 2 \lor 3, 4, \bar{1} \lor 2 \lor 4</td>
<td>no</td>
<td>by Restart</td>
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<td>1, 2 \lor 3, 4, \bar{1} \lor 2 \lor 4</td>
<td>\bar{1} \lor 3 \lor 4</td>
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<tr>
<td>1 4 • 2</td>
<td>(1, 2 \lor 3, 4)</td>
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<td>Decide</td>
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<tr>
<td>1 4 • 2</td>
<td>(1, 2 \lor 3, 4)</td>
<td>(\overline{1} \lor 2 \lor 4)</td>
<td>T-Conflict</td>
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<tr>
<td>1 4 • 2</td>
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Modeling the Very Lazy Theory Approach

\[ g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d \]

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Modeling the Very Lazy Theory Approach

\[
g(a) = c \land \overbrace{f(g(a)) \neq f(c)}^{2} \lor \overbrace{g(a) = d}^{3} \land c \neq d
\]

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fail
Modeling the Very Lazy Theory Approach

\[ g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d \]

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\[
g(a) = c \quad \wedge \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \wedge \quad c \neq d
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The very lazy approach can be improved considerably with

- An *on-line* SAT engine,
  which can accept new input clauses on the fly

- an *incremental and explicating* $T$-solver,
  which can
  1. check the $T$-satisfiability of $M$ as it is extended and
  2. identify a small $T$-unsatisfiable subset of $M$ once $M$ becomes $T$-unsatisfiable
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A Better Lazy Approach

\[
g(a) = c 
\quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{1} \quad \lor \quad \underbrace{g(a) = d}_{2} \quad \lor \quad \underbrace{c \neq d}_{4}
\]

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A Better Lazy Approach

\[ g(a) = c \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\text{1}} \quad \lor \quad \underbrace{g(a) = d}_{\text{3}} \quad \lor \quad \underbrace{c \neq d}_{\text{4}} \]

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A Better Lazy Approach

\[ g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d \]

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\[ g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d \]

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\[ g(a) = c \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\text{1}} \quad \lor \quad \underbrace{g(a) = d}_{\text{2}} \quad \land \quad c \neq d \quad \text{\text{4}} \]

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\[ g(a) = c \quad \land \quad \begin{cases} \frac{f(g(a)) \neq f(c)}{2} \lor \quad \begin{cases} g(a) = d \quad \land \quad c \neq d \end{cases} 
\end{cases} \]

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<td>4</td>
<td></td>
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</tr>
<tr>
<td>1</td>
<td>4 \bullet 2</td>
<td></td>
<td>Decide</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>T-Conflict</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>Backjump</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>Propagate</td>
</tr>
<tr>
<td>fail</td>
<td></td>
<td></td>
<td>T-Conflict</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Fail</td>
</tr>
</tbody>
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Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment $M$, apply **Conflict**
2. If $M$ is $T$-unsatisfiable, apply **$T$-Conflict**
3. Apply **Fail** or **Explain+Learn+Backjump** as appropriate
4. Apply **Propagate**
5. Apply **Decide**

**Note:** Depending on the cost of checking the $T$-satisfiability of $M$, Step (2) can be applied with lower frequency or priority.
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Theory Propagation

With $T$-Conflict as the only theory rule, the theory solver is used just to validate the choices of the SAT engine.

With $T$-Propagate and $T$-Explain, it can also be used to guide the engine’s search [Tin02]

$T$-Propagate

\[
\begin{align*}
 & l \in \text{Lit}(F) \quad M \models_T l \quad l, \bar{l} \notin M \\
 & \quad M := M \downarrow l
\end{align*}
\]

$T$-Explain

\[
\begin{align*}
 & C = l \lor D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n <_M \bar{l} \\
 & \quad C := l_1 \lor \cdots \lor l_n \lor D
\end{align*}
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\textbf{T-Propagate} \quad l \in \text{Lit}(F) \quad M \models T l \quad l, \overline{l} \notin M

\quad M := M l

\textbf{T-Explain} \quad C = l \lor D \quad \overline{l}_1, \ldots, \overline{l}_n \models T \overline{l} \quad \overline{l}_1, \ldots, \overline{l}_n <_M \overline{l}

\quad C := l_1 \lor \cdots \lor l_n \lor D
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\[
g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d
\]

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<tr>
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<th>C</th>
<th>rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1, 2 ∨ 3, 4</td>
<td>no</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1, 2 ∨ 3, 4</td>
<td>no</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2 ∨ 3</td>
<td>no</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2 ∨ 3</td>
<td>2 ∨ 3</td>
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<tr>
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Note: $T$-propagation eliminates search altogether in this case.
No applications of Decide are needed.
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\[
g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d
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</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1, 2 ∨ 3, 4</td>
<td>no</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1, 2 ∨ 3, 4</td>
<td>no \ by Propagate⁺</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>no \ by T-Propagate (1 \models_T 2)</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>no \ by T-Propagate (1, 4 \models_T 3)</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>2 ∨ 3 \ by Conflict</td>
</tr>
<tr>
<td>fail</td>
<td>4</td>
<td>2</td>
<td>by Fail</td>
</tr>
</tbody>
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Theory Propagation Example

\[
g(a) = c \quad \land \quad \overline{f(g(a))} \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d
\]

<table>
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<tr>
<th>M</th>
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<th>rule</th>
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<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>(1, \overline{2} \lor 3, \overline{4})</td>
<td>no</td>
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<tr>
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<td>4</td>
<td>(1, \overline{2} \lor 3, \overline{4})</td>
<td>no</td>
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<tr>
<td>1</td>
<td>4</td>
<td>(1, 2 \lor 3, 4)</td>
<td>no</td>
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<tr>
<td>1</td>
<td>4</td>
<td>(1, 2 \lor 3, 4)</td>
<td>(\bot)</td>
</tr>
<tr>
<td>fail</td>
<td></td>
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Theory Propagation Example

\[ g(a) = c \quad \wedge \quad f(g(a)) \neq f(c) \quad \vee \quad g(a) = d \quad \wedge \quad c \neq d \]

\[ \begin{array}{cccc}
M & F & C & \text{rule} \\
1 4 & 1, \overline{2} \lor 3, \overline{4} & \text{no} & \text{by Propagate}^+ \\
1 4 2 & 1, \overline{2} \lor 3, \overline{4} & \text{no} & \text{by } T\text{-Propagate } (1 \models_T 2) \\
1 4 2 3 & 1, \overline{2} \lor 3, \overline{4} & \text{no} & \text{by } T\text{-Propagate } (1, \overline{4} \models_T 3) \\
1 4 2 3 & \overline{2} \lor 3 & \text{by Conflict} \\
\text{fail} & & \text{by Fail} \\
\end{array} \]

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Theory Propagation Example

\[
g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d
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<tr>
<th>M</th>
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<tbody>
<tr>
<td>(1)</td>
<td>(\frac{2}{4})</td>
<td>(1, \frac{2}{3}, \frac{4}{3})</td>
<td>no</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>(\frac{3}{4})</td>
<td>no</td>
</tr>
<tr>
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<td>(\frac{1}{2})</td>
<td>(\frac{3}{4})</td>
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</tr>
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<td>(\frac{1}{2})</td>
<td>(\frac{3}{4})</td>
<td>no</td>
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Theory Propagation Example

\[ g(a) = c \quad \land \quad f(g(a)) \neq f(c) \quad \lor \quad g(a) = d \quad \land \quad c \neq d \]

\[ \begin{array}{cccc}
    & M & F & C & \text{rule} \\
1 & 1, 2 \lor 3, 4 & \text{no} & & \\
1 4 & 1, 2 \lor 3, 4 & \text{no} & \text{by Propagate}^+ \\
1 4 2 & 1, 2 \lor 3, 4 & \text{no} & \text{by } T\text{-Propagate} \ (1 \models_T 2) \\
1 4 2 3 & 1, 2 \lor 3, 4 & \text{no} & \text{by } T\text{-Propagate} \ (1, 4 \models_T 3) \\
1 4 2 3 & 1, 2 \lor 3, 4 & \bar{2} \lor 3 & \text{by Conflict} \\
\end{array} \]

\[ \text{Note: } T\text{-propagation eliminates search altogether in this case} \]
\[ \text{no applications of Decide are needed} \]
**Theory Propagation Example**

\[
g(a) = \begin{cases} c \land f(g(a)) \neq f(c) \lor g(a) = d \land c \neq d \\
1 \quad 2 \quad 3 \quad 4
\end{cases}
\]

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<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>no</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>no by <strong>Propagate</strong>^+</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>no by <strong>T-Propagate</strong> (1 \models_T 2)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>no by <strong>T-Propagate</strong> (1, 4 \models_T 3)</td>
</tr>
<tr>
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<td>3</td>
<td>4</td>
<td>(2 \lor 3) by <strong>Conflict</strong></td>
</tr>
<tr>
<td>fail</td>
<td></td>
<td></td>
<td></td>
<td>by <strong>Fail</strong></td>
</tr>
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**Note:** \(T\)-propagation eliminates search altogether in this case
no applications of **Decide** are needed
At the core, current lazy SMT solvers are implementations of the transition system with rules

1. Propagate, Decide, Conflict, Explain, Backjump, Fail

2. $T$-Conflict, $T$-Propagate, $T$-Explain

3. Learn, Forget, Restart

**Basic DPLL Modulo Theories** $\overset{\text{def}}{=} (1) + (2)$

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Correctness

Updated terminology:

**Irreducible state**: state to which no Basic DPLL MT rules apply

**Execution**: sequence of transitions allowed by the rules and starting with $M = \emptyset$ and $C = \text{no}$

**Exhausted execution**: execution ending in an irreducible state

**Proposition** (Soundness) For every exhausted execution starting with $F = F_0$ and ending with fail, the clause set $F_0$ is $T$-unsatisfiable.

**Proposition** (Completeness) For every exhausted execution starting with $F = F_0$ and ending with $C = \text{no}$, $F_0$ is $T$-satisfiable; specifically, $M$ is $T$-satisfiable and $M \models_T F_0$. 
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**Proposition** (Termination) Every execution in which

(a) **Learn/Forget** are applied only *finitely many times* and

(b) **Restart** is applied with *increased periodicity*

is finite.

**Lemma** Every exhausted execution ends with either \( C = \text{no} \) or *fail*.

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The approach formalized so far can be implemented with a simple architecture named $\text{DPLL}(T)$ \cite{GHN+04, NOT06}

$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$
DPLL($T$) Architecture

The approach formalized so far can be implemented with a simple architecture named \textbf{DPLL($T$)} \cite{GN04,NOT06}

\textbf{DPLL($T$)} = \textbf{DPLL($X$)} engine + $T$-solver

\textbf{DPLL($X$)}:

- \textbf{Very similar to a SAT solver}, enumerates Boolean models
- \textbf{Not allowed}: pure literal, blocked literal detection, ...
- \textbf{Required}: incremental addition of clauses
- \textbf{Desirable}: partial model detection
The approach formalized so far can be implemented with a simple architecture named \textbf{DPLL}(T) [GHN+04, NOT06]

$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine } + T\text{-solver}$$

\textit{T}-solver:

- Checks the \textit{T}-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of \textit{T}-unsatisfiability/propagation
- Must be incremental and backtrackable
For certain theories, determining that a set $M$ is $T$-unsatisfiable requires reasoning by cases.

**Example:** $T$ = the theory of arrays.

$$M = \{ r(w(a, i, x), j) \neq x, \ r(w(a, i, x), j) \neq r(a, j) \}$$

1. $i = j$) Then, $r(w(a, i, x), j) = x$. Contradiction with 1.

2. $i \neq j$) Then, $r(w(a, i, x), j) = r(a, j)$. Contradiction with 2.

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Reasoning by Cases in Theory Solvers

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**Conclusion:** $M$ is $T$-unsatisfiable
A complete $T$-solver reasons by cases via (internal) case splitting and backtracking mechanisms.

An alternative is to lift case splitting and backtracking from the $T$-solver to the SAT engine.

**Basic idea:** encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them [BNOT06]

**Possible benefits:**

- All case-splitting is coordinated by the SAT engine
- Only have to implement case-splitting infrastructure in one place
- Can learn a wider class of lemmas
Case Splitting

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Basic Scenario:

$$M = \{\ldots, s = r(w(a, i, t), j), \ldots\}$$

- Main SMT module: “Is $M$ $T$-unsatisfiable?”
- $T$-solver: “I do not know yet, but it will help me if you consider these theory lemmas:

$$s = s' \land i = j \rightarrow s = t, \quad s = s' \land i \neq j \rightarrow s = r(a, j)$$"
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To model the generation of theory lemmas for case splits, add the rule

\[ T\text{-Learn} \]

\[ \models_T \exists \mathbf{v}(l_1 \lor \cdots \lor l_n) \quad l_1, \ldots, l_n \in L_S \quad \mathbf{v} \text{ vars not in } F \]

\[ F := F \cup \{l_1 \lor \cdots \lor l_n\} \]

where \( L_S \) is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of \( L_S \))

**Note:** For many theories with a theory solver, there exists an appropriate finite \( L_S \) for every input \( F \). The set \( L_S \) does not need to be computed explicitly.
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The set \( L_S \) does not need to be computed explicitly
Now we can relax the requirement on the theory solver:

When $M \models_p F$, it must either

- determine whether $M \models_T \bot$
- generate a new clause by $T$-Learn containing at least one literal of $L_S$ undefined in $M$

The $T$-solver is required to determine whether $M \models_T \bot$ only if all literals in $L_S$ are defined in $M$

Note: In practice, to determine if $M \models_T \bot$, the $T$-solver only needs a small subset of $L_S$ to be defined in $M$
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- determine whether \( M \models_T \bot \) or
- generate a new clause by \textbf{\( T\)-Learn} containing at least one literal of \( L_S \) undefined in \( M \)

The \( T \)-solver is \textbf{required} to determine whether \( M \models_T \bot \) only if all literals in \( L_S \) are defined in \( M \)

\textbf{Note}: In practice, to determine if \( M \models_T \bot \), the \( T \)-solver only needs a small subset of \( L_S \) to be defined in \( M \)
Example — Theory of Finite Sets

\[ F : \quad x = y \cup z \quad \land \quad y \neq \emptyset \vee x = z \]

<table>
<thead>
<tr>
<th>M</th>
<th>( x = y \cup z )</th>
<th>F</th>
<th>( x = y \cup z ) • ( y = \emptyset ) ( x \neq z )</th>
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\( T \)-solver can make the following deductions at this point:

\[ e \in x \quad \implies \quad e \in y \cup z \quad \implies \quad e \in y \quad \implies \quad e \in \emptyset \quad \implies \quad \bot \]

This enables an application of \( T \)-Conflict with clause

\[ x \neq y \cup z \quad \land \quad y \neq \emptyset \quad \land \quad x = z \quad \land \quad e \notin x \quad \land \quad e \in z \]
Example — Theory of Finite Sets

\[ F: \quad x = y \cup z \quad \land \quad y \not= \emptyset \lor x \not= z \]

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<td>[ x = y \cup z \quad \land \quad y = \emptyset ]</td>
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<td>[ x = y \cup z \quad \land \quad y = \emptyset \quad \land \quad x \not= z ]</td>
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<td>by Decide</td>
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\[ T \]-solver can make the following deductions at this point:

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This enables an application of \( T \)-Conflict with clause

\[ x \not= y \cup z \lor y \not= \emptyset \lor x = z \lor e \not\in x \lor e \in z \]
Example — Theory of Finite Sets

\( F : \ x = y \cup z \ \land \ y \neq \emptyset \lor x \neq z \)

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<td>( F )</td>
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This enables an application of $T$-Conflict with clause

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\[ F : \quad x = y \cup z \land y \neq \emptyset \lor x \neq z \]

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This enables an application of \( T \)-Conflict with clause

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Example — Theory of Finite Sets

\[ F : x = y \cup z \land y \neq \emptyset \lor x \neq z \]

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Example — Theory of Finite Sets

\[ F : x = y \cup z \land y \neq \emptyset \lor x \neq z \]

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Correctness Results

Correctness results can be extended to the new rule.

**Soundness:** The new $T$-Learn rule maintains satisfiability of the clause set.

**Completeness:** As long as the theory solver can decide $\mathcal{M} \models_T \bot$ when all literals in $\mathcal{L}_S$ are determined, the system is still complete.

**Termination:** The system terminates under the same conditions as before. Roughly:

- Any lemma is (re)learned only finitely many times
- **Restart** is applied with increased periodicity
Combining Theories
Need for Combining Theories and Solvers

**Recall:** Many applications give rise to formulas like:

\[ a \approx b + 2 \land A \approx \text{write}(B, a + 1, 4) \land \\
(\text{read}(A, b + 3) \approx 2 \lor f(a - 1) \neq f(b + 1)) \]

Solving that formula requires reasoning over

- the theory of linear arithmetic \((T_{\text{LA}})\)
- the theory of arrays \((T_{\text{A}})\)
- the theory of uninterpreted functions \((T_{\text{UF}})\)

**Question:** Given solvers for each theory, can we combine them modularly into one for \(T_{\text{LA}} \cup T_{\text{A}} \cup T_{\text{UF}}\)?

Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]
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Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]
Consider the following set of literals over $T_{LRA} \cup T_{UF}$ ($T_{LRA}$, linear real arithmetic):

\[
\begin{align*}
  f(f(x) - f(y)) &= a \\
  f(0) &> a + 2 \\
  x &= y
\end{align*}
\]

**First step:** *purify* literals so that each belongs to a single theory
Consider the following set of literals over $T_{\text{LRA}} \cup T_{\text{UF}}$ ($T_{\text{LRA}}$, linear real arithmetic):

\[
\begin{align*}
  f(f(x) - f(y)) & = \ a \\
  f(0) & > \ a + 2 \\
  x & = \ y
\end{align*}
\]

**First step:** *purify* literals so that each belongs to a single theory
Motivating Example (Convex Case)

Consider the following set of literals over $T_{LRA} \cup T_{UF}$ ($T_{LRA}$, linear real arithmetic):

\begin{align*}
f(f(x) - f(y)) &= a \\
f(0) &> a + 2 \\
x &= y
\end{align*}

**First step:** *purify* literals so that each belongs to a single theory

\begin{align*}
f(f(x) - f(y)) = a \implies f(e_1) &= a \\
e_1 = f(x) - f(y) \implies f(e_1) &= a \\
e_1 &= e_2 - e_3 \\
e_2 &= f(x) \\
e_3 &= f(y)
\end{align*}
Consider the following set of literals over $T_{\text{LRA}} \cup T_{\text{UF}}$ ($T_{\text{LRA}}$, linear real arithmetic):

\[
\begin{align*}
f(f(x) - f(y)) &= a \\
f(0) &= a + 2 \\
x &= y
\end{align*}
\]

**First step:** *purify* literals so that each belongs to a single theory

\[
\begin{align*}
f(0) &> a + 2 \implies f(e_4) > a + 2 \implies f(e_4) = e_5 \\
e_4 &= 0 \\
e_5 &> a + 2
\end{align*}
\]
Motivating Example (Convex Case)

**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

\[
\begin{array}{ll}
L_1 & L_2 \\
\hline
f(e_1) = a & e_2 - e_3 = e_1 \\
f(x) = e_2 & e_4 = 0 \\
f(y) = e_3 & e_5 > a + 2 \\
f(e_4) = e_5 & e_2 = e_3 \\
x = y & a = e_5 \\
e_1 = e_4 & \\
\end{array}
\]

$L_1 \models_{UF} e_2 = e_3 \quad L_2 \models_{LRA} e_1 = e_4$

$L_1 \models_{UF} a = e_5$

**Third step:** check for satisfiability locally

Report unsatisfiable
Motivating Example (Convex Case)

**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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$L_1 \models_{UF} e_2 = e_3$ \quad $L_2 \models_{LRA} e_1 = e_4$

Third step: check for satisfiability locally

Report unsatisfiable
Motivating Example (Convex Case)

**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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$L_1 \models_{UF} a = e_5$

**Third step:** check for satisfiability locally

Report unsatisfiable
Motivating Example (Convex Case)

**Second step:** exchange entailed *interface equalities*, equalities over shared constants \(e_1, e_2, e_3, e_4, e_5, a\)

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\[L_1 \models_{\text{UF}} e_2 = e_3\]

\[L_2 \models_{\text{LRA}} e_1 = e_4\]

\[L_1 \models_{\text{UF}} a = e_5\]

**Third step:** check for satisfiability locally

\[L_1 \models_{\text{UF}} \bot\]

Report unsatisfiable
Motivating Example (Convex Case)

**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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\end{array}
\]

$L_1 \models_{UF} e_2 = e_3 \quad L_2 \models_{LRA} e_1 = e_4$

$L_1 \models_{UF} a = e_5$

**Third step:** check for satisfiability locally

$L_1 \models_{UF} L \quad \text{Report unsatisfiable}$
**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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$L_1 \models_{UF} a = e_5$

**Third step:** check for satisfiability locally

$L_1 \models_{UF} \bot$  \quad Report unsatisfiable

$L_2 \models_{LRA} \bot$
**Motivating Example (Convex Case)**

**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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$L_1 \models_{UF} e_2 = e_3$ \quad $L_2 \models_{LRA} e_1 = e_4$

$L_1 \models_{UF} a = e_5$

**Third step:** check for satisfiability locally

$L_1 \models_{UF} \bot$ \quad Report unsatisfiable

$L_2 \models_{LRA} \bot$
Motivating Example (Convex Case)

**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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$L_1 \models_{UF} e_2 = e_3$  \quad  L_2 \models_{LRA} e_1 = e_4$

$L_1 \models_{UF} a = e_5$

**Third step:** check for satisfiability locally

$L_1 \models_{UF} \bot$

$L_2 \models_{LRA} \bot$

Report unsatisfiable
**Motivating Example (Convex Case)**

**Second step:** exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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$L_1 \models_{UF} e_2 = e_3$  \quad L_2 \models_{LRA} e_1 = e_4$

$L_1 \models_{UF} a = e_5$

**Third step:** check for satisfiability locally

$L_1 \models_{UF} \perp$

$L_2 \models_{LRA} \perp$  \quad Report unsatisfiable
Consider the following unsatisfiable set of literals over $T_{\text{LIA}} \cup T_{\text{UF}}$ ($T_{\text{LIA}}$, linear integer arithmetic):

\begin{align*}
1 \leq & \quad x \quad \leq 2 \\
 f(1) & = \quad a \\
 f(2) & = \quad f(1) + 3 \\
 a & = \quad b + 2
\end{align*}

First step: purify literals so that each belongs to a single theory.
Consider the following unsatisfiable set of literals over $T_{\text{LIA}} \cup T_{\text{UF}}$ ($T_{\text{LIA}}$, linear integer arithmetic):

\begin{align*}
1 & \leq x \leq 2 \\
f(1) & = a \\
f(2) & = f(1) + 3 \\
a & = b + 2
\end{align*}

**First step:** *purify* literals so that each belongs to a single theory
Motivating Example (Non-convex Case)

Consider the following unsatisfiable set of literals over $T_{\text{LIA}} \cup T_{\text{UF}}$ ($T_{\text{LIA}}$, linear integer arithmetic):

\[
\begin{align*}
1 \leq x & \leq 2 \\
f(1) & = a \\
f(2) & = f(1) + 3 \\
a & = b + 2
\end{align*}
\]

First step: *purify* literals so that each belongs to a single theory

\[
f(1) = a \implies f(e_1) = a \\
e_1 = 1
\]
Motivating Example (Non-convex Case)

Consider the following unsatisfiable set of literals over $T_{LIA} \cup T_{UF}$ ($T_{LIA}$, linear integer arithmetic):

$$1 \leq x \leq 2$$
$$f(1) = a$$
$$f(2) = f(1) + 3$$
$$a = b + 2$$

First step: purify literals so that each belongs to a single theory

$$f(2) = f(1) + 3 \implies e_2 = 2$$
$$f(e_2) = e_3$$
$$f(e_1) = e_4$$
$$e_3 = e_4 + 3$$
**Second step:** exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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<tr>
<td>$a = b + 2$</td>
<td>$f(e_1) = e_4$</td>
</tr>
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<td>$e_2 = 2$</td>
<td>$x = e_1$</td>
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<tr>
<td>$e_3 = e_4 + 3$</td>
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Second step: exchange entailed interface equalities over shared constants \( x, e_1, a, b, e_2, e_3, e_4 \)

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<td>( x \leq 2 )</td>
<td>( f(x) = b )</td>
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<td>( e_1 = 1 )</td>
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<tr>
<td>( a = b + 2 )</td>
<td>( f(e_1) = e_4 )</td>
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<tr>
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<tr>
<td>( e_3 = e_4 + 3 )</td>
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<td>( a = e_4 )</td>
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No more entailed equalities, but \( L_1 \models_{\text{LIA}} x = e_1 \lor x = e_2 \)
Motivating Example (Non-convex Case)

**Second step:** exchange entailed *interface equalities* over shared constants \(x, e_1, a, b, e_2, e_3, e_4\)

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Consider each case of \(x = e_1 \lor x = e_2\) separately
Motivating Example (Non-convex Case)

**Second step:** exchange entailed interface equalities over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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Case 1) $x = e_1$
Motivating Example (Non-convex Case)

**Second step:** exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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$L_2 \models_{UF} a = b$, which entails $\bot$ when sent to $L_1$
Motivating Example (Non-convex Case)

**Second step:** exchange entailed *interface equalities* over shared constants \(x, e_1, a, b, e_2, e_3, e_4\)

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Motivating Example (Non-convex Case)

**Second step:** exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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<tr>
<td>$x = e_2$</td>
<td></td>
</tr>
</tbody>
</table>

Case 2) $x = e_2$
Second step: exchange entailed interface equalities over shared constants $x, e_1, a, b, e_2, e_3, e_4$

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq x$</td>
<td>$f(e_1) = a$</td>
</tr>
<tr>
<td>$x \leq 2$</td>
<td>$f(x) = b$</td>
</tr>
<tr>
<td>$e_1 = 1$</td>
<td>$f(e_2) = e_3$</td>
</tr>
<tr>
<td>$a = b + 2$</td>
<td>$f(e_1) = e_4$</td>
</tr>
<tr>
<td>$e_2 = 2$</td>
<td>$x = e_2$</td>
</tr>
<tr>
<td>$e_3 = e_4 + 3$</td>
<td></td>
</tr>
<tr>
<td>$a = e_4$</td>
<td></td>
</tr>
<tr>
<td>$x = e_2$</td>
<td></td>
</tr>
</tbody>
</table>
Motivating Example (Non-convex Case)

**Second step:** exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

<table>
<thead>
<tr>
<th>$L_1$</th>
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</tr>
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<tbody>
<tr>
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<tr>
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<td></td>
</tr>
<tr>
<td>$x = e_2$</td>
<td></td>
</tr>
</tbody>
</table>

$L_2 \models_{UF} e_3 = b$, which entails $\bot$ when sent to $L_1$
The Nelson-Oppen Method

• For \( i = 1, 2 \), let \( T_i \) be a first-order theory of signature \( \Sigma_i \) (set of function and predicate symbols in \( T_i \) other than \( = \))

• Let \( T = T_1 \cup T_2 \)

• Let \( C \) be a finite set of free constants (i.e., not in \( \Sigma_1 \cup \Sigma_2 \))

We consider only input problems of the form

\[
L_1 \cup L_2
\]

where each \( L_i \) is a finite set of ground (i.e., variable-free) \((\Sigma_i \cup C)\)-literals

Note: Because of purification, there is no loss of generality in considering only ground \((\Sigma_i \cup C)\)-literals
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The Nelson-Oppen Method

Bare-bones, non-deterministic, non-incremental version

[Opp80, Rin96, TH96]:

Input: \(L_1 \cup L_2\) with \(L_i\) finite set of ground \((\Sigma_i \cup C)\)-literals

Output: sat or unsat

1. Guess an arrangement \(A\), i.e., a set of equalities and disequalities over \(C\) such that
   \[c = d \in A \text{ or } c \neq d \in A\] for all \(c, d \in C\)

2. If \(L_i \cup A\) is \(T_i\)-unsatisfiable for \(i = 1\) or \(i = 2\), return unsat

3. Otherwise, return sat
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\]

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Correctness of the NO Method

**Proposition** (Termination) The method is terminating.

(Trivially, because there is only a finite number of arrangements to guess)

**Proposition** (Soundness) If the method returns unsat for every arrangement, the input is \((T_1 \cup T_2)\)-unsatisfiable.

(Because satisfiability in \((T_1 \cup T_2)\) is always preserved)

**Proposition** (Completeness) If \(\Sigma_1 \cap \Sigma_2 = \emptyset\) and \(T_1\) and \(T_2\) are stably infinite, when the method returns sat for some arrangement, the input is \((T_1 \cup T_2)\)-is satisfiable.
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Stably Infinite Theories

**Def.** A theory $T$ is *stably infinite* iff every quantifier-free $T$-satisfiable formula is satisfiable in an *infinite* model of $T$

Many interesting theories are stably infinite:

- Theories of an *infinite structure* (e.g., integer arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
- Convex theories (e.g., EUF, linear real arithmetic)

**Def.** A theory $T$ is *convex* iff, for any set $L$ of literals

$$L \models_T s_1 = t_1 \lor \cdots \lor s_n = t_n \implies L \models_T s_i = t_i \text{ for some } i$$

**Note:** With convex theories, arrangements do not need to be guessed—they can be computed by (theory) propagation
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- Some equational/Horn theories

The Nelson-Oppen method has been extended to some classes of non-stably infinite theories [TZ05, RRZ05, JB10]
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Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$.

How can we integrate all of them \textit{cooperatively} into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?
SMT Solving with *Multiple* Theories

Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$.

How can we integrate all of them *cooperatively* into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

**Quick Solution:**

1. Combine $S_1, \ldots, S_n$ with Nelson-Oppen into a theory solver for $T$

2. Build a DPLL($T$) solver as usual
Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$.

How can we integrate all of them cooperatively into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

**Better Solution** [Bar02, BBC+05b, BNOT06]:

1. Extend DPLL($T$) to DPLL($T_1, \ldots, T_n$)
2. **Lift Nelson-Oppen to the DPLL($X_1, \ldots, X_n$) level**
3. Build a DPLL($T_1, \ldots, T_n$) solver
Modeling DPLL($T_1, \ldots, T_n$) Abstractly

- Let $n = 2$, for simplicity

- Let $T_i$ be of signature $\Sigma_i$ for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$

- Let $C$ be a set of free constants

- Assume wlog that each input literal has signature $(\Sigma_1 \cup C)$ or $(\Sigma_2 \cup C)$ (no mixed literals)

- Let $M|_i \overset{\text{def}}{=} \{(\Sigma_i \cup C)\text{-literals of } M \text{ and their complement}\}$

- Let $I(M) \overset{\text{def}}{=} \{c = d \mid c, d \text{ occur in } C, M|_1 \text{ and } M|_2\} \cup \{c \neq d \mid c, d \text{ occur in } C, M|_1 \text{ and } M|_2\}$

  (interface literals)
Abstract DPLL Modulo Multiple Theories

**Propagate, Conflict, Explain, Backjump, Fail** (unchanged)

**Decide**

\[
l \in \text{Lit}(F) \cup \text{I}(M) \quad l, \bar{l} \notin M
\]

\[
M := M \cdot l
\]

Only change: decide on interface equalities as well

**T-Propagate**

\[
l \in \text{Lit}(F) \cup \text{I}(M) \quad i \in \{1, 2\} \quad M \models T_i l \quad l, \bar{l} \notin M
\]

\[
M := M \cdot l
\]

Only change: propagate interface equalities as well, but reason locally in each \(T_i\)
Propagate, Conflict, Explain, Backjump, Fail (unchanged)

Decide
\[
\frac{l \in \text{Lit}(F) \cup I(M) \quad l, \bar{l} \notin M}{M := M \bullet l}
\]

Only change: decide on interface equalities as well

\[
\begin{align*}
T\text{-Propagate} & \quad l \in \text{Lit}(F) \cup I(M) \quad i \in \{1, 2\} \quad M \models_{T_i} l \quad l, \bar{l} \notin M \\
\quad & \quad M := M \bullet l
\end{align*}
\]

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Propagate, Conflict, Explain, Backjump, Fail (unchanged)

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\[ l \in \text{Lit}(F) \cup I(M) \quad i \in \{1, 2\} \quad M \models T_i l \quad l, \overline{l} \not\in M \]
\[ M := M l \]

Only change: propagate interface equalities as well, but reason locally in each \( T_i \)
Abstract DPLL Modulo Multiple Theories

\( T\)-Conflict

\[
C = \text{no} \quad l_1, \ldots, l_n \in M \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}
\]

\[
C := \bar{l}_1 \lor \cdots \lor \bar{l}_n
\]

\( T\)-Explain

\[
C = l \lor D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n <_{M} \bar{l}
\]

\[
C := \bar{l}_1 \lor \cdots \lor \bar{l}_n \lor \lor D
\]

Only change: reason locally in each \( T_i \)

I-Learn

\[
\models_{T_i} \bar{l}_1 \lor \cdots \lor \bar{l}_n \quad l_1, \ldots, l_n \in M_{i} \cup I(M) \quad i \in \{1, 2\}
\]

\[
F := F \cup \{l_1 \lor \cdots \lor l_n\}
\]

New rule: for entailed disjunctions of interface literals
Abstract DPLL Modulo Multiple Theories

**$T$-Conflict**

\[ C = \text{no} \quad l_1, \ldots, l_n \in M \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\} \]

\[ C := \overline{l}_1 \lor \cdots \lor \overline{l}_n \]

**$T$-Explain**

\[ C = l \lor D \quad \overline{l}_1, \ldots, \overline{l}_n \models_{T_i} \overline{l} \quad i \in \{1, 2\} \quad \overline{l}_1, \ldots, \overline{l}_n \models_M \overline{l} \]

\[ C := l_1 \lor \cdots \lor l_n \lor D \]

*Only change:* reason locally in each $T_i$

**I-Learn**

\[ \models_{T_i} l_1 \lor \cdots \lor l_n \quad l_1, \ldots, l_n \in M|_i \cup I(M) \quad i \in \{1, 2\} \]

\[ F := F \cup \{l_1 \lor \cdots \lor l_n\} \]

*New rule:* for entailed disjunctions of interface literals
Example — Convex Theories

\[ F := \begin{align*}
&\begin{array}{ll}
0 & f(e_1) = a \\
1 & f(x) = e_2 \\
2 & f(y) = e_3 \\
3 & f(e_4) = e_5 \\
4 & x = y \\
5 & e_2 - e_3 = e_1 \\
6 & e_4 = 0 \\
7 & e_5 > a + 2 \\
8 & e_2 = e_3 \\
9 & e_1 = e_4 \\
10 & a = e_5
\end{array}
\end{align*} \]

<table>
<thead>
<tr>
<th>M</th>
<th>F</th>
<th>C</th>
<th>rule</th>
</tr>
</thead>
</table>
| 0 1 2 3 4 5 6 7 | F | no | by Propagate
| 0 1 2 3 4 5 6 7 8 | F | no | by T-Propagate \((1, 2, 4 \models UF 8)\)
| 0 1 2 3 4 5 6 7 8 9 | F | no | by T-Propagate \((5, 6, 8 \models LRA 9)\)
| 0 1 2 3 4 5 6 7 8 9 10 | F | no | by T-Propagate \((0, 3, 9 \models UF 10)\)
| 0 1 2 3 4 5 6 7 8 9 10 | F | 7 \lor 10 | by T-Conflict \((7, 10 \models LRA \bot)\)
| fail | | | by Fail |
Example — Convex Theories

\[
F := \begin{cases} 
0 & f(e_1) = a \\
1 & f(x) = e_2 \\
2 & f(y) = e_3 \\
3 & f(e_4) = e_5 \\
4 & x = y \\
5 & e_2 - e_3 = e_1 \\
6 & e_4 = 0 \\
7 & e_5 > a + 2 \\
8 & e_2 = e_3 \\
9 & e_1 = e_4 \\
10 & a = e_5 
\end{cases}
\]

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<th>C</th>
<th>rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>no</td>
<td>by \textbf{Propagate}^+</td>
<td></td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7</td>
<td>( F )</td>
<td>no</td>
<td>by \textbf{T-Propagate} ((1, 2, 4 \models_{\text{UF}} 8))</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8</td>
<td>( F )</td>
<td>no</td>
<td>by \textbf{T-Propagate} ((5, 6, 8 \models_{\text{LRA}} 9))</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8 9</td>
<td>( F )</td>
<td>no</td>
<td>by \textbf{T-Propagate} ((0, 3, 9 \models_{\text{UF}} 10))</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8 9 10</td>
<td>( F )</td>
<td>(7 \lor 10)</td>
<td>by \textbf{T-Conflict} ((7, 10 \models_{\text{LRA}} \bot))</td>
</tr>
<tr>
<td>fail</td>
<td>( F )</td>
<td>by \textbf{Fail}</td>
<td></td>
</tr>
</tbody>
</table>
Example — Convex Theories

\[ F := \begin{align*}
& f(e_1) = a \land e_2 - e_3 = e_1 \land e_4 = 0 \land x = y \\
& f(x) = e_2 \land e_5 > a + 2 \\
& f(y) = e_3 \\
& f(e_4) = e_5 \land e_1 = e_4 \land a = e_5
\end{align*} \]

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<th>F</th>
<th>C</th>
<th>rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2 3 4 5 6 7</td>
<td>no</td>
<td></td>
<td>by Propagate (^{+})</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8</td>
<td>no</td>
<td></td>
<td>by T-Propagate ((1, 2, 4 \models_{\text{UF}} 8))</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8 9</td>
<td>no</td>
<td></td>
<td>by T-Propagate ((5, 6, 8 \models_{\text{LRA}} 9))</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8 9 10</td>
<td>no</td>
<td></td>
<td>by T-Propagate ((0, 3, 9 \models_{\text{UF}} 10))</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8 9 10</td>
<td>7 \lor 10</td>
<td>fail</td>
<td>by T-Conflict ((7, 10 \models_{\text{LRA}} \bot))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>by Fail</td>
</tr>
</tbody>
</table>
Example — Convex Theories

\[ F := \left\{ \begin{array}{l}
  0 \quad f(e_1) = a \\
  1 \quad f(x) = e_2 \\
  2 \quad f(y) = e_3 \\
  3 \quad f(e_4) = e_5 \\
  4 \quad x = y
\end{array} \right. \land \left\{ \begin{array}{l}
  5 \quad e_2 - e_3 = e_1 \\
  6 \quad e_4 = 0 \\
  7 \quad e_5 > a + 2 \\
  8 \quad e_2 = e_3 \\
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\end{array} \right. \]

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<th>C</th>
<th>rule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>yes</td>
<td>no</td>
<td>by Propagate^+</td>
</tr>
<tr>
<td>01234567F</td>
<td>no</td>
<td>by T-Propagate (1, 2, 4 \models_{UF} 8)</td>
<td></td>
</tr>
<tr>
<td>012345678F</td>
<td>no</td>
<td>by T-Propagate (5, 6, 8 \models_{LRA} 9)</td>
<td></td>
</tr>
<tr>
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<td>yes</td>
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Example — Convex Theories

\[ F := \begin{align*}
0 & \quad f(e_1) = a \land e_2 - e_3 = e_1 \\
1 & \quad f(x) = e_2 \land e_4 = 0 \\
2 & \quad f(y) = e_3 \land e_5 > a + 2 \\
3 & \quad f(e_4) = e_5 \land x = y \\
4 & \quad e_2 = e_3 \land e_1 = e_4 \land a = e_5
\end{align*} \]

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<tr>
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<td>F</td>
<td>no</td>
<td>by \text{T-Propagate} \ (1, 2, 4 \models_{\text{UF}} 8)</td>
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<td></td>
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<tr>
<td></td>
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<td>no</td>
<td>by \text{T-Propagate} \ (0, 3, 9 \models_{\text{UF}} 10)</td>
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<tr>
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<td>no</td>
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<td>F</td>
<td>no</td>
<td>by \text{Fail}</td>
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</table>
Example — Convex Theories

\[ F := \begin{align*}
\forall & 0 & f(e_1) = a \land e_2 - e_3 = e_1 \\
\forall & 5 & f(x) = e_2 \land e_4 = 0 \\
\forall & 6 & f(y) = e_3 \land e_5 > a + 2 \\
\forall & 7 & f(e_4) = e_5 \land x = y \\
\forall & 8 & e_2 = e_3 \\
\forall & 9 & e_1 = e_4 \\
\forall & 10 & a = e_5
\end{align*} \]

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<tr>
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<td>10</td>
<td>by Fail</td>
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</table>
Example — Convex Theories

\[ F := \ \begin{align*}
  & f(e_1) = a \land e_2 - e_3 = e_1 \land e_2 - e_3 = e_1 \\
  & f(x) = e_2 \land e_4 = 0 \land e_5 > a + 2 \\
  & f(y) = e_3 \land e_5 > a + 2 \\
  & f(e_4) = e_5 \land x = y \\
\end{align*} \]

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<td>(F)</td>
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<td>by Propagate (+)</td>
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<td>no</td>
<td>by (T)-Propagate ((1, 2, 4 \vdash_{UF} 8))</td>
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<tr>
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<td>(F)</td>
<td>no</td>
<td>by (T)-Propagate ((5, 6, 8 \vdash_{LRA} 9))</td>
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<tr>
<td>0 1 2 3 4 5 6 7 8 9</td>
<td>(F)</td>
<td>no</td>
<td>by (T)-Propagate ((0, 3, 9 \vdash_{UF} 10))</td>
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<tr>
<td>0 1 2 3 4 5 6 7 8 9 10</td>
<td>(F)</td>
<td>7 \lor 10</td>
<td>by (T)-Conflict ((7, 10 \vdash_{LRA} \bot))</td>
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</tbody>
</table>
Example — Convex Theories

\[ F := \begin{align*}
  f(e_1) &= a, \\
  f(x) &= e_2, \\
  e_2 - e_3 &= e_1, \\
  e_4 &= 0, \\
  e_5 &= a + 2, \\
  x &= y. 
\end{align*} \]

\[ e_2 = e_3, \quad e_1 = e_4, \quad a = e_5 \]

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<td>0 1 2 3 4 5 6 7</td>
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<td>by Propagate^+</td>
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<tr>
<td>0 1 2 3 4 5 6 7 8</td>
<td>F</td>
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<td>by T-Propagate (1, 2, 4 \models_{UF} 8)</td>
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<tr>
<td>0 1 2 3 4 5 6 7 8 9</td>
<td>F</td>
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<td>by T-Propagate (5, 6, 8 \models_{LRA} 9)</td>
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<tr>
<td>0 1 2 3 4 5 6 7 8 9 10</td>
<td>F</td>
<td>no</td>
<td>by T-Propagate (0, 3, 9 \models_{UF} 10)</td>
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<tr>
<td>0 1 2 3 4 5 6 7 8 9 10</td>
<td>F</td>
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<td>by T-Conflict (7, 10 \models_{LRA} \bot)</td>
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<tr>
<td>fail</td>
<td>F</td>
<td>no</td>
<td>by Fail</td>
</tr>
</tbody>
</table>
Example — Non-convex Theories

\[ F := \begin{cases} 
0 & f(e_1) = a \land f(x) = b \land 1 \leq x \leq 2 \land e_1 = 1 \land a = b + 2 \land e_2 = 2 \land e_3 = e_4 + 3 \\
1 & a = e_4 \\
2 & x = e_1 \\
3 & x = e_2 \\
4 & a = b \\
\end{cases} \]

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<td>by Propagate (exercise)</td>
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<td>by Fail</td>
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Example — Non-convex Theories

\[ F := \begin{align*}
 f(e_1) &= a \wedge f(x) &= b \wedge f(e_2) &= e_3 \wedge f(e_1) &= e_4 \wedge \\
 1 \leq x &\leq 2 \wedge e_1 &= 1 \wedge a &= b + 2 \wedge e_2 &= 2 \wedge e_3 &= e_4 + 3 
\end{align*} \]

\[ a = e_4 \quad x = e_1 \quad x = e_2 \quad a = b \]

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<td>0 \ldots 9 10</td>
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<td>by T-Propagate $(0, 3 \models_{UF} 10)$</td>
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<td>by I-Learn $(\models_{LIA} \overline{4}$ vortex $\overline{5}$ vortex $\overline{11}$ vortex $\overline{12}$)</td>
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<td>0 \ldots 9 10 11 13</td>
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<td>by Decide</td>
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<tr>
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<td>by T-Propagate $(0, 1, 11 \models_{UF} 13)$</td>
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<td>by Backjump</td>
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<tr>
<td>0 \ldots 9 10 13 11</td>
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<tr>
<td>\ldots</td>
<td>\ldots</td>
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<td>by Fail</td>
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Example — Non-convex Theories

\[ F := f(e_1) = a \land f(x) = b \land e_1 = 1 \land a = b + 2 \land e_2 = 2 \land e_3 = e_4 + 3 \]

\[
\begin{align*}
& a = e_4 \\
& x = e_1 \\
& x = e_2 \\
& a = b
\end{align*}
\]

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<tr>
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\[ \ldots \]
Example — Non-convex Theories

\[ F := f(e_1) = a \land f(x) = b \land f(e_2) = e_3 \land f(e_1) = e_4 \land 1 \leq x \leq 2 \land e_1 = 1 \land a = b + 2 \land e_2 = 2 \land e_3 = e_4 + 3 \]

\[ a = e_4 \quad x = e_1 \quad x = e_2 \quad a = b \]

\[ \begin{array}{ccc}
    M & F & C \mid \text{rule} \\
    \hline
    0 & \ldots & 9 & F & \text{no} & \text{by Propagate}^+ \\
    0 & \ldots & 9 & 10 & F & \text{no} & \text{by } T\text{-Propagate } (0, 3 \models_U 10) \\
    0 & \ldots & 9 & 10 & 1 & F & \text{no} & \text{by } T\text{-Propagate } (0, 1, 11 \models_U 13) \\
    0 & \ldots & 9 & 10 & 11 & 13 & F & \text{no} & \text{by } T\text{-Conflict } (7, 13 \models_U \bot) \\
    0 & \ldots & 9 & 10 & 11 & 13 & 11 & F & \text{no} & \text{by } T\text{-Propagate } (0, 1, 13 \models_U 11) \\
    0 & \ldots & 9 & 10 & 11 & 12 & F & \text{no} & \text{by Propagate } (exercise) \\
    \ldots & \ldots \ldots & \text{fail} & \ldots & \text{by Fail} \\
\end{array} \]
**Example — Non-convex Theories**

\[ F := \begin{align*}
\begin{array}{lllllllll}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
f(e_1) = a & f(x) = b & f(e_2) = e_3 & f(e_1) = e_4 & 1 \leq x & x \leq 2 & e_1 = 1 & a = b + 2 & e_2 = 2 & e_3 = e_4 + 3 \\
\end{array}
\end{align*} \]

\[ a = e_4 \quad x = e_1 \quad x = e_2 \quad a = b \]

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<tr>
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<td>by <strong>I-Learn</strong> ( (\vdash_{\text{LIA}} 4 \lor 5 \lor 11 \lor 12) )</td>
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<tr>
<td>no</td>
<td>by <strong>Decide</strong></td>
<td></td>
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<tr>
<td>no</td>
<td>by <strong>T-Propagate</strong> ( (0, 1, 11) \vdash_{\text{UF}} 13 )</td>
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<td>7 \lor 13</td>
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<td>by <strong>Backjump</strong></td>
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<tr>
<td>no</td>
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<td>by Fail</td>
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...
Example — Non-convex Theories

\[
F := \begin{cases} 
0 & \text{if } f(e_1) = a \\
1 & \text{if } f(x) = b \\
2 & \text{if } f(e_2) = e_3 \\
3 & \text{if } f(e_1) = e_4 \\
1 \leq x & \text{if } x \leq 2 \\
4 & \text{if } e_1 = 1 \\
5 & \text{if } a = b + 2 \\
6 & \text{if } e_2 = 2 \\
7 & \text{if } e_3 = e_4 + 3 \\
\end{cases}
\]

\[
\begin{align*}
a &= e_4 & x &= e_1 \\
10 & & 11 & & 12 & & 13 \\
& & & & & & & \\
\end{align*}
\]

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<th>rule</th>
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<tr>
<td>0 \ldots 9 10 11</td>
<td>F, ( \overline{4} \lor \overline{5} \lor 11 \lor 12 )</td>
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<td>by \textbf{I-Learn} ( (\models_{\text{LIA}} \overline{4} \lor \overline{5} \lor 11 \lor 12) )</td>
</tr>
<tr>
<td>0 \ldots 9 10 11</td>
<td>F, ( \overline{4} \lor \overline{5} \lor 11 \lor 12 )</td>
<td>no</td>
<td>by \textbf{Decide}</td>
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<td>by \textbf{T-Conflict} ( (7, 13 \models_{\text{UF}} \bot) )</td>
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<td></td>
<td></td>
<td>no</td>
<td>by \text{Backjump}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>no</td>
<td>by \textbf{T-Propagate} ( (0, 1, 13 \models_{\text{UF}} \bot) )</td>
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<tr>
<td></td>
<td></td>
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<td>by \textbf{Propagate} ( ) (exercise)</td>
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<td>by \text{Fail}</td>
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Example — Non-convex Theories

\[
F := \begin{align*}
f(e_1) &= a \land f(x) &= b \land f(e_2) = e_3 \land f(e_1) = e_4 \land \\
1 \leq x & \leq 2 \land e_1 = 1 \land a &= b + 2 \land e_2 = 2 \land e_3 = e_4 + 3 \\
a &= e_4 & x &= e_1 & x &= e_2 & a &= b \\
10 & & 11 & & 12 & & 13
\end{align*}
\]

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<th>rule</th>
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<td>0 \ldots 9</td>
<td>F</td>
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<tr>
<td>0 \ldots 9 10</td>
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<td>by T-Conflict ((7, 13 \models_{UF} \bot))</td>
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<td>by T-Propagate ((0, 1, 13 \models_{UF} \bot))</td>
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<tr>
<td>\ldots</td>
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<td>\ldots</td>
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<td></td>
<td>by Fail</td>
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</table>
Example — Non-convex Theories

\[ F := \{ \begin{align*}
\text{if } e_1 & \text{ then } a, \\
\text{if } x & \leq 2, \\
\text{if } e_2 & = e_3, \\
\text{if } a & = b + 2, \\
\text{if } e_2 & = 2, \\
\text{if } e_3 & = e_4 + 3.
\end{align*} \]

\[ a = e_4 \quad x = e_1 \quad x = e_2 \quad a = b \]

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<tr>
<td>0</td>
<td>( F )</td>
<td>no</td>
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<td>no</td>
<td>by Decide</td>
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Example — Non-convex Theories

\[ F := \begin{align*}
&f(e_1) = a \land f(x) = b \land f(e_2) = e_3 \land f(e_1) = e_4 \land \\
&1 \leq x \leq 2 \land e_1 = 1 \land a = b + 2 \land e_2 = 2 \land e_3 = e_4 + 3 \\
&\quad a = e_4 \quad x = e_1 \quad x = e_2 \quad a = b
\end{align*} \]

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<td>F</td>
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</tr>
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</table>
Example — Non-convex Theories

\[ F := \begin{array}{l}
\begin{aligned}
& f(e_1) = a \land f(x) = b \land f(e_2) = e_3 \land f(e_1) = e_4 \\
& 1 \leq x \land x \leq 2 \land e_1 = 1 \land a = b + 2 \land e_2 = 2 \land e_3 = e_4 + 3
\end{aligned}
\end{array} \]

\[ a = e_4 \quad x = e_1 \quad x = e_2 \quad a = b \]

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<td>no</td>
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<td>no</td>
<td>by Backjump</td>
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<td>no</td>
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Example — Non-convex Theories

\[ F := f(e_1) = a \land f(x) = b \land f(e_2) = e_3 \land f(e_1) = e_4 \land 1 \leq x \leq 2 \land e_1 = 1 \land a = b + 2 \land e_2 = 2 \land e_3 = e_4 + 3 \]

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... by Fail
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