Guessing Game

I'm thinking of a research area where:

- Algorithms have recently improved by *orders of magnitude*
- Computers solve tasks *better than* humans
- Computers solve tasks *without help* from humans
- *Big investments* are being made by the government and industry, including companies like:
  - Amazon, Apple, Facebook, Google, Intel, Microsoft
- It can be described using two letters; first one is *A*

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```
(assert (forall ((lambda Real)) (let ((v_17 (+ x4 (* 60 lambda))) (v_11 (not bool.b19)) (v_10 (not bool.b18)) (v_6 (not bool.b17)) (v_8 (not bool.b21)) (v_3 (not bool.b23)) (v_
9 (not bool.b20)) (v_7 (not bool.b22))) (let ((v_4 (and v_9 v_7))) (let ((v_2 (not v_4)) (v_13 (and v_10 v_11))) (let ((v_12 (not v_13)) (v_14 (and v_9 v_13))) (let ((v_15 (and
 v_8 v_14))) (let ((v_16 (and v_7 v_15)) (v_40 (<= (* 1 v_17) 4820))) (let ((v_72 (not v_40)) (v_99 (not bool.b24)) (v_127 (+ x3 (* (/ (- 1) 20) lambda)))) (let ((v_1 (* 1 v_12
7))) (let ((v_0 (+ v_1 (* (/ 1 1200) v_17)))) (let ((v_96 (<= v_0 (/ 20 3))) (v_5 (<= v_1 0))) (let ((v_42 (not v_5)) (v_137 (<= v_1 40))) (let ((v_19 (not v_137)) (v_21 (* (-
1) v_17))) (let ((v_43 (<= v_21 (- 4100)))) (let ((v_38 (not v_43)) (v_20 (not (<= v_1 33)))) (let ((v_35 (and bool.b17 (not (and v_19 (and v_38 v_20)))))) (let ((v_18 (not v_3
5)) (v_45 (<= v_21 (- 4500)))) (let ((v_78 (not v_45)) (v_39 (<= v_21 (- 4910)))) (let ((v_94 (not v_39))) (let ((v_57 (not (and bool.b19 (not (and v_19 (and v_20 v_94)))))) (
let ((v_22 (not (and (and v_18 (not (and bool.b18 (not (and v_19 (and v_20 v_78)))))) v_57)))) (let ((v_32 (and bool.b23 v_22))) (let ((v_36 (not v_32)) (v_29 (and bool.b22 v_2
2))) (let ((v_33 (not v_29)) (v_26 (and bool.b21 v_22))) (let ((v_30 (not v_26)) (v_24 (and bool.b20 v_22))) (let ((v_27 (not v_24)) (v_23 (and bool.b18 v_22)) (v_47 (and bool.
b19 v_22))) (let ((v_28 (and (not v_23) (not v_47)))) (let ((v_25 (not v_28)) (v_31 (and v_27 v_28))) (let ((v_34 (and v_30 v_31))) (let ((v_37 (and v_33 v_34)) (v_123 (<= (+ v
_1 (* (/ 1 15) v_17)) (/ 964 3)))) (let ((v_121 (not v_123))) (let ((v_101 (and v_5 v_121))) (let ((v_67 (not v_101)) (v_49 (and v_38 v_35)) (v_48 (not (and bool.b19 v_39)))) (
let ((v_50 (and v_48 v_36)) (v_41 (and v_5 v_24))) (let ((v_59 (not (and v_40 v_41)))) (let ((v_52 (and v_33 v_59)) (v_58 (not (and v_72 v_41)))) (let ((v_54 (and v_30 v_58)) (
v_44 (not (and bool.b17 v_43))) (v_56 (and bool.b18 v_45))) (let ((v_46 (not v_56))) (let ((v_129 (and v_44 v_46))) (let ((v_61 (and v_44 (not (and v_129 v_23)))) (v_77 (and v_
46 (not (and v_47 (and v_46 v_48)))))) (let ((v_55 (and (not (and v_42 v_24)) (and v_61 v_77)))) (let ((v_53 (and v_54 v_55))) (let ((v_51 (and v_52 v_53))) (let ((v_68 (not (a
nd v_49 (not (and v_50 v_51))))) (v_69 (not v_49)) (v_62 (and bool.b24 (not (and v_18 v_57))))) (let ((v_60 (not (and v_56 (not v_62))))) (let ((v_65 (and v_60 (not (and v_24 (
and v_58 (and v_59 v_60)))))) (v_63 (not (and v_56 v_62)))) (let ((v_64 (and v_63 (not (and v_47 (and v_48 v_63)))))) (let ((v_66 (not (and v_61 v_64))) (v_79 (not v_61)) (v_70
 (not v_64)) (v_71 (not v_65))) (let ((v_83 (not (and v_39 v_70)))) (let ((v_90 (and v_50 v_83)) (v_73 (and v_5 v_71))) (let ((v_76 (not (and v_40 v_73)))) (let ((v_88 (and v_5
2 v_76)) (v_74 (not (and v_72 v_73)))) (let ((v_86 (and v_54 v_74)) (v_81 (and bool.b17 v_45))) (let ((v_75 (not (and v_99 v_81)))) (let ((v_84 (and v_75 (not (and v_71 (and v_
74 (and v_75 v_76)))))) (v_80 (and v_78 v_79))) (let ((v_108 (not (and (not v_77) v_80))) (v_93 (not v_80)) (v_82 (not (and bool.b24 v_81)))) (let ((v_95 (and v_82 (not (and v_
70 (and v_82 v_83)))))) (let ((v_87 (and v_93 v_95))) (let ((v_85 (not v_87)) (v_92 (not v_84)) (v_89 (and v_84 v_87))) (let ((v_91 (and v_86 v_89)) (v_105 (and v_42 v_92))) (l
et ((v_107 (not v_105)) (v_130 (and v_94 (not v_95)))) (let ((v_106 (not v_130))) (let ((v_120 (and v_22 (and v_107 (and (and v_69 v_93) v_106)))) (v_125 (+ v_1 (* (/ 1 20) v_1
7)))) (let ((v_126 (not (<= v_125 241))) (v_97 (not (and bool.b24 v_56)))) (let ((v_98 (not (and v_97 (not (and bool.b19 (and v_48 v_97))))))) (let ((v_114 (and (not (and v_39
(not (and v_82 (not (and v_98 (and (not (and v_39 v_98)) v_82))))))) v_90)) (v_100 (not (and v_99 v_56)))) (let ((v_102 (not (and v_100 (not (and bool.b20 (and (and v_100 (not
(and v_40 (and bool.b20 v_5)))) (not (and bool.b20 v_101)))))))) (let ((v_103 (and v_5 v_102))) (let ((v_104 (and v_5 (not (and v_75 (not (and v_102 (and (not (and v_72 v_103))
) (and (not (and v_40 v_103)) v_75)))))))) (let ((v_112 (and (not (and v_40 v_104)) v_88)) (v_110 (and (not (and v_72 v_104)) v_86)) (v_111 (and v_93 v_106))) (let ((v_109 (no
t v_111)) (v_118 (not v_110)) (v_113 (and v_107 v_111)) (v_117 (not v_112))) (let ((v_115 (and v_110 v_113)) (v_116 (not v_114))) (let ((v_133 (not (and v_5 (and v_40 (not (and
 v_68 (not (and v_69 (not (and (not (and v_114 (not (and (not (and v_112 (not (and (not (and v_110 (not (and (not (and v_105 v_109)) (not (and v_107 (not (and v_108 v_109))))))
)) (not (and v_118 (not v_113)))))))) (not (and v_117 (not v_115))))))) (not (and v_116 (not (and v_112 v_115)))))))))))))) (v_119 (not (and v_114 v_112))) (v_122 (<= v_1 20)))
(let ((v_135 (and v_122 (and v_121 v_105))) (v_124 (and v_122 (and v_123 v_105)))) (let ((v_143 (not v_124)) (v_128 (not (<= (* (- 1) v_127) (- 20)))) (v_138 (and bool.b17 v_38
)) (v_140 (and v_78 (not (and v_44 (not (and bool.b18 v_129))))))) (let ((v_134 (and (and (not (and v_128 v_138)) (not (and v_128 v_140))) (not (and v_128 v_130)))) (v_132 (and
 v_107 v_112))) (let ((v_131 (not v_132))) (let ((v_144 (and (not (and v_134 (not (and (not (and v_42 (not (and v_106 (and v_93 (and v_69 (and (not (and v_110 (not (and (not (a
nd v_114 (not (and v_131 (not (and v_105 v_117)))))) (not (and v_116 v_131)))))) (not (and v_118 (not (and v_114 v_132))))))))))) v_133)))) (not (and v_67 (not v_134))))) (v_13
6 (+ v_1 (* (/ 3 20) v_17))) (v_141 (<= v_1 45))) (let ((v_139 (and v_141 v_20)) (v_142 (not v_141))) (or (or (exists ((lambdaprime Real)) (let ((v_145 (* 1 (+ x3 (* (/ (- 1) 2
0) lambdaprime))))) (let ((v_146 (not (<= v_145 40))) (v_148 (* (- 1) (+ x4 (* 60 lambdaprime)))) (v_147 (not (<= v_145 33)))) (and (and (<= 0 lambdaprime) (<= lambdaprime lamb
da)) (not (and (and (not (and bool.b17 (not (and v_146 (and (not (<= v_148 (- 4100))) v_147))))) (not (and bool.b18 (not (and v_146 (and v_147 (not (<= v_148 (- 4500))))))))) (
not (and bool.b19 (not (and v_146 (and v_147 (not (<= v_148 (- 4910))))))))))))) (< lambda 0)) (and (not (and v_96 (and (not (<= v_0 (/ 241 60))) (and (not (and v_42 (not (and
 v_11 (and v_10 (and v_6 (and (not (and v_8 (not (and (not (and v_3 (not (and v_2 (not (and bool.b20 bool.b22)))))) (not (and bool.b23 v_2)))))) (not (and bool.b21 (not (and v_
3 v_4))))))))))) (not (and v_5 (and (not (and (not (and v_6 (not (and (not (and v_3 (not (and (not (and v_7 (not (and (not (and v_8 (not (and (not (and v_9 (not (and v_12 (not
(and bool.b18 bool.b19)))))) (not (and bool.b20 v_12)))))) (not (and bool.b21 (not v_14)))))) (not (and bool.b22 (not v_15)))))) (not (and bool.b23 (not v_16))))))) (not (and
 bool.b17 (not (and v_3 v_16)))))) v_40))))))) (not (and (and (not (and v_18 (not (and (not (and v_36 (not (and (not (and v_33 (not (and (not (and v_30 (not (and (not (and v_27
 (not (and v_25 (not (and bool.b19 v_23)))))) (not (and v_24 v_25)))))) (not (and v_26 (not v_31))))))) (not (and v_29 (not v_34))))))) (not (and v_32 (not v_37))))))) (not (an
d v_35 (not (and v_36 v_37))))) (not (and v_67 (not (and (and v_68 (not (and v_69 (not (and (not (and (not v_50) (not v_51))) (not (and v_50 (not (and (not (and (not v_52) (not
 v_53))) (not (and v_52 (not (and (not (and (not v_54) (not v_55))) (not (and v_54 (not (and (not (and v_65 (not (and v_66 (not (and v_79 v_70)))))) (not (and v_71 v_66)))))))
)))))))))) (not (and v_67 (not (and (and v_68 (not (and v_69 (not (and (not (and v_90 (not (and (not (and v_88 (not (and (not (and v_86 (not (and (not (and v_84 (not (and v_10
8 v_85)))) (not (and v_92 v_85)))))) (not (and (not v_86) (not v_89))))))) (not (and (not v_88) (not v_91))))))) (not (and (not v_90) (not (and v_88 v_91))))))))) (not (and (no
t (and v_120 (and v_96 (and v_126 (and v_133 (not (and v_42 (not (and v_69 (and v_107 (and v_106 (and v_93 (and (not (and v_110 (not (and v_119 (not (and v_116 v_117)))))) (not
 (and v_118 v_119)))))))))))))))) (not (and (not v_120) (and (not (and (not v_135) (not (and (not (and v_67 v_143)) (not (and v_124 (not (and (<= v_125 400) (and v_126 v_144))))
```

A (Very) Brief History of Automated Reasoning

Philosophers have long dreamed of machines that can reason. The pursuit of this dream has occupied some of the best minds and led both to great acheivements and great disappointments.

~1700

Leibniz – mechanized human reasoning

1928

Hilbert

Entscheidungs-problem

1936

Church – lamda calculus

Turing – reduction halting problem

1954

Davis – decision procedure for Presburger arithmetic

Automated Reasoning: A Failure?

- At the turn of the century, automated reasoning was still considered by many to be impractical for most real-world applications
- Interesting problems appeared to be beyond the reach of automated methods because of decidability and complexity barriers
- The dream of *Hilbert*'s mechanized mathematics or *Leibniz*'s calculating machine was believed by many to be simply unattainable

Princeton, c. 2000

- *Chaff SAT solver*: orders of magnitude faster than previous SAT solvers
- *Important observation*: many real-world problems do not exhibit worst-case theoretical performance

Palo Alto, c. 2001

- Idea: combine fast new SAT solvers with decision procedures for decidable first-order theories
- *SVC*, *CVC* solvers (Stanford); *ICS*, *Yices* solvers (SRI)
- *Satisfiability Modulo Theories* (SMT) was born

SMT solvers

SMT solvers: *general-purpose* logic engines

- Given condition X, is it possible for Y to happen
- X and Y are expressed in a *rich logical language*
    - First-order logic
    - Domain-specific reasoning
        - arithmetic, arrays, bit-vectors, data types, etc.

SMT solvers are *changing the way people solve problems*

- Instead of building a *special-purpose* solver
- *Translate* into a logical formula and use an SMT solver
- Not only easier, *often better*

SMT solvers

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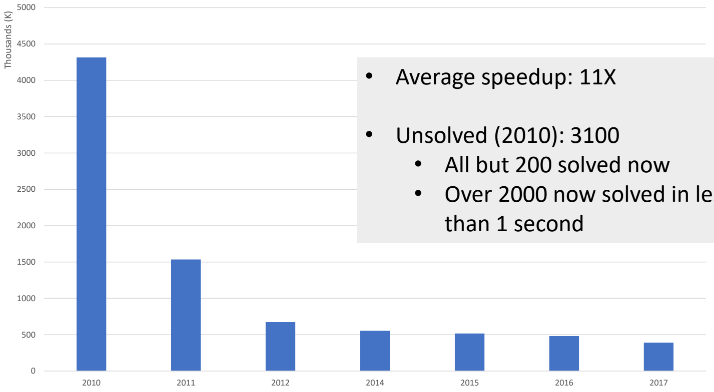
Evolution of SMT solving

- Total time on QF_BV benchmarks (virtual best)



- Average speedup: 11X
- Unsolved (2010): 3100
  - All but 200 solved now
  - Over 2000 now solved in less than 1 second

Zelkova

Security Policy

```
(allow,
 principal     : *,
 action        : getObject,
 resource      : cs240/*,
 condition     : (StringEquals, aws:sourceVpc, vpc-111bbb222),
                 (StringLike, s3:prefix, cs240/Exam*))
```

SMT Encoding

a = “getObject” ∧ r = “cs240/*” ∧ vpcExists ∧
vpc = “vpc-111bbb222” ∧ s3PrefixExists ∧
“cs240/Exam” prefixOf s3Prefix

SMT Solvers (cvc5 and z3)

Strings and RegExp	Bitvectors	Arithmetic

Satisfiability Modulo Theories

Clark Barrett, Stanford University

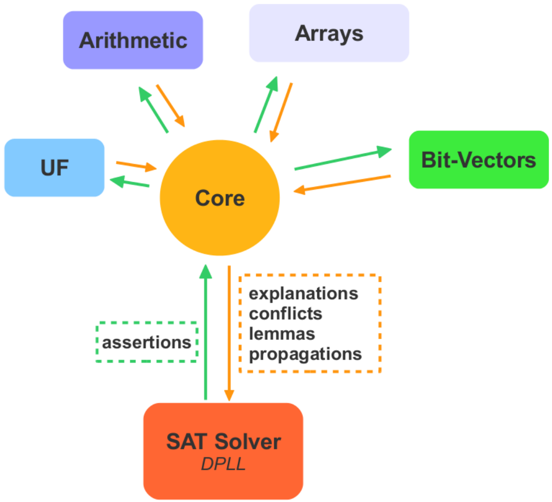
SAT + SMT Winter School, Dec 15, 2023

Acknowledgments: Many thanks to Cesare Tinelli and Albert Oliveras for contributing some of the material used in these slides.

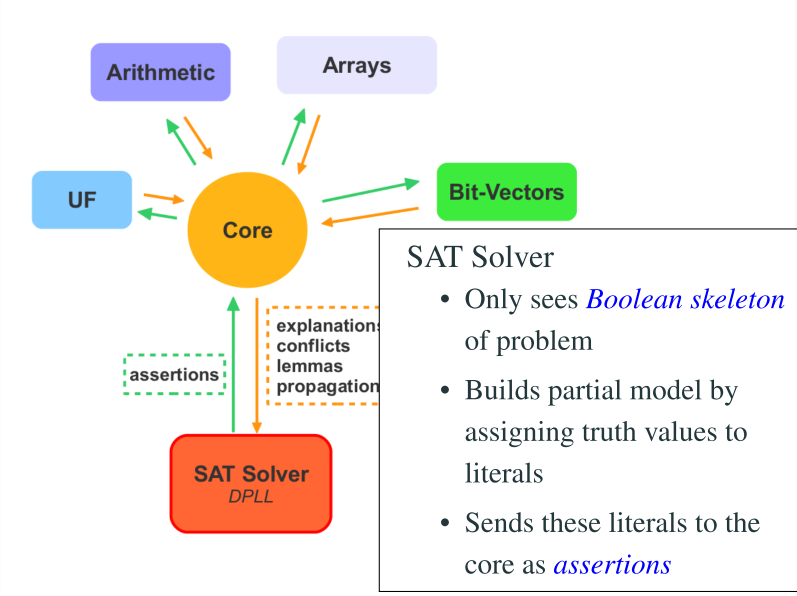
Disclamer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

Introduction

SMT Solvers


Arithmetic
Arrays
UF
Core
Bit-Vectors
assertions
explanations
conflicts
lemmas
propagations
SAT Solver
DPLL

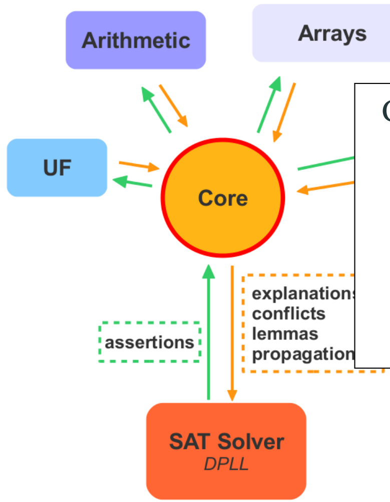
SMT Solvers



SAT Solver

- Only sees *Boolean skeleton* of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as *assertions*

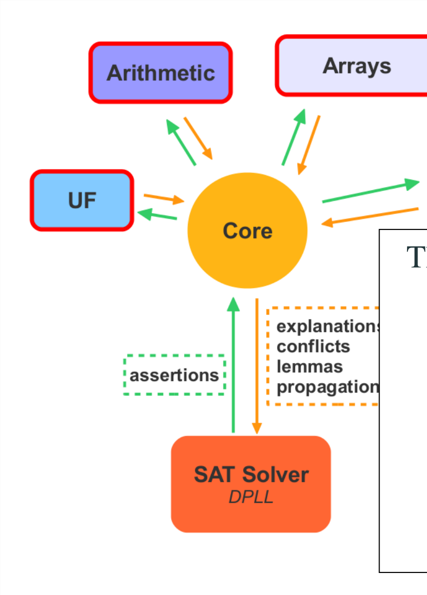
SMT Solvers



Core

- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

SMT Solvers



Theory Solvers

- Decide T-satisfiability of a conjunction of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation

Theory Solvers

Theory Solvers

Given a theory T, a *Theory Solver* for T takes as input a set Φ of literals and determines whether Φ is T-satisfiable.

Φ is T-satisfiable iff there is some model M of T such that each formula in Φ holds in M.

Theories of Interest: UF

Equality ($=$) with Uninterpreted Functions [NO80, BD94, NO07]

Typically used to abstract unsupported constructs, e.g.,

- non-linear multiplication in arithmetic
- ALUs in circuits

Example: The formula

$$a * (|b| + c) = d \;\; \wedge \;\; b * (|a| + c) \neq d \;\; \wedge \;\; a = b$$

is unsatisfiable, but no arithmetic reasoning is needed

if we abstract it to

$$mul(a, add(abs(b), c)) = d \;\; \wedge \;\; mul(b, add(abs(a), c)) \neq d \;\; \wedge \;\; a = b$$

it is still unsatisfiable

Theories of Interest: Arithmetic

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ [BBC+05a]
- Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ [LM05]
- Linear arithmetic, e.g., $2x - 3y + 4z \leqslant 5$ [DdM06]
- Non-linear arithmetic, e.g., $2xy + 4xz^2 - 5y \leqslant 10$ [BLNM+09, ZM10, JdM12]

Theories of Interest: Arrays

Used in software verification and hardware verification (for memories) [SBDL01, BNO+08a, dMB09]

Two interpreted function symbols read and write

Axiomatized by:

- $\forall a\, \forall i\, \forall v\ \mathrm{read}(\mathrm{write}(a, i, v), i) = v$
- $\forall a\, \forall i\, \forall j\, \forall v\ i \neq j \rightarrow \mathrm{read}(\mathrm{write}(a, i, v), j) = \mathrm{read}(a, j)$

Sometimes also with *extensionality* :

- $\forall a\, \forall b\ (\forall i\, \mathrm{read}(a, i) = \mathrm{read}(b, i)\ \rightarrow a = b)$

Is the following set of literals satisfiable in this theory?

$$\mathrm{write}(a, i, x) \neq b,\ \mathrm{read}(b, i) = y,\ \mathrm{read}(\mathrm{write}(b, i, x), j) = y,\ a = b,\ i = j$$

Theories of Interest: Bit vectors

Useful both in hardware and software verification [BCF+07, BB09, HBJ+14]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- *String-like*: concat, extract, ...
- *Logical*: bit-wise not, or, and, ...
- *Arithmetic*: add, subtract, multiply, ...
- *Comparison*: $<, >, \ldots$

Is this formula satisfiable over bit vectors of size 3?

$$a[1:0] \neq b[1:0] \;\wedge\; (a \mid b) = c \;\wedge\; c[0] = 0 \;\wedge\; a[1] + b[1] = 0$$

Implementing a Theory Solver: Difference Logic

We consider a simple example: difference logic.

In *difference logic*, we are interested in the satisfiability of a conjunction of arithmetic atoms.

Each atom is of the form $x - y \bowtie c$, where x and y are variables, c is a numeric constant, and $\bowtie \in \{=, <, \leqslant, >, \geqslant\}$.

The variables can range over either the *integers* (QF_IDL) or the *reals* (QF_RDL).

Difference Logic

The first step is to rewrite everything in terms of $\leqslant$:

Difference Logic

The first step is to rewrite everything in terms of $\leqslant$:

- $x - y = c \quad \Longrightarrow \quad x - y \leqslant c \,\wedge\, x - y \geqslant c$

Difference Logic

The first step is to rewrite everything in terms of $\leqslant$:

- $x - y = c \quad \implies \quad x - y \leqslant c \;\wedge\; x - y \geqslant c$
- $x - y \geqslant c \quad \implies \quad y - x \leqslant -c$

Difference Logic

The first step is to rewrite everything in terms of $\leqslant$:

- $x - y = c \quad \Longrightarrow \quad x - y \leqslant c \,\wedge\, x - y \geqslant c$
- $x - y \geqslant c \quad \Longrightarrow \quad y - x \leqslant -c$
- $x - y > c \quad \Longrightarrow \quad y - x < -c$

Difference Logic

The first step is to rewrite everything in terms of $\leqslant$:

- $x - y = c \quad \Longrightarrow \quad x - y \leqslant c \ \wedge \ x - y \geqslant c$
- $x - y \geqslant c \quad \Longrightarrow \quad y - x \leqslant -c$
- $x - y > c \quad \Longrightarrow \quad y - x < -c$
- $x - y < c \quad \Longrightarrow \quad x - y \leqslant c - 1$ (integers)

Difference Logic

The first step is to rewrite everything in terms of $\leqslant$:

- $x - y = c \quad \Longrightarrow \quad x - y \leqslant c \ \wedge \ x - y \geqslant c$
- $x - y \geqslant c \quad \Longrightarrow \quad y - x \leqslant -c$
- $x - y > c \quad \Longrightarrow \quad y - x < -c$
- $x - y < c \quad \Longrightarrow \quad x - y \leqslant c - 1$ (integers)
- $x - y < c \quad \Longrightarrow \quad x - y \leqslant c - \delta$ (reals)

Difference Logic

Now we have a conjunction of literals, all of the form $x - y \leqslant c$.

From these literals, we form a weighted directed graph with a vertex for each variable.

For each literal $x - y \leqslant c$, there is an edge $x \xrightarrow{c} y$.

The set of literals is satisfiable iff there is no cycle for which the sum of the weights on the edges is negative.

There are a number of efficient algorithms for detecting negative cycles in graphs.

Difference Logic Example

$$x - y = 5 \ \wedge \ z - y \geqslant 2 \ \wedge \ z - x > 2 \ \wedge \ w - x = 2 \ \wedge \ z - w < 0$$

Difference Logic Example

$$x - y = 5 \;\wedge\; z - y \geqslant 2 \;\wedge\; z - x > 2 \;\wedge\; w - x = 2 \;\wedge\; z - w < 0$$

$$x - y = 5$$
$$z - y \geqslant 2$$
$$z - x > 2$$
$$w - x = 2$$
$$z - w < 0$$

Difference Logic Example

$$x - y = 5 \ \land \ z - y \geqslant 2 \ \land \ z - x > 2 \ \land \ w - x = 2 \ \land \ z - w < 0$$

$$\begin{aligned} x - y &= 5 \\ z - y &\geqslant 2 \\ z - x &> 2 \quad \Rightarrow \\ w - x &= 2 \\ z - w &< 0 \end{aligned}$$

Difference Logic Example

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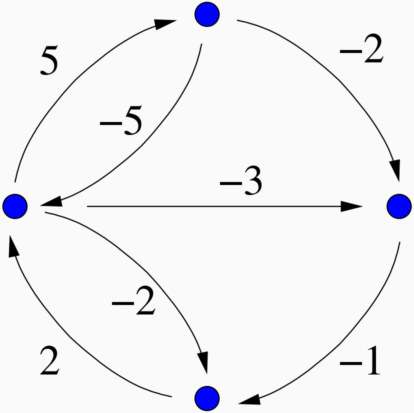
$$\begin{array}{lcl} x - y = 5 & & x - y \leqslant 5 \wedge y - x \leqslant -5 \\ z - y \geqslant 2 & & y - z \leqslant -2 \\ z - x > 2 & \Rightarrow & x - z \leqslant -3 \\ w - x = 2 & & w - x \leqslant 2 \wedge x - w \leqslant -2 \\ z - w < 0 & & z - w \leqslant -1 \end{array}$$

Difference Logic Example



DPLL(T): Combining T-Solvers with SAT

Satisfiability Modulo a Theory T

Def. A formula is *(un)satisfiable in* a theory T, or T-*(un)satisfiable*, if there is a (no) model of T that satisfies it

Note: The T-satisfiability of quantifier-free formulas is decidable iff the T-satisfiability of conjunctions/sets of literals is decidable

(Convert the formula in DNF and check if any of its disjuncts is T-sat)

Problem: In practice, dealing with Boolean combinations of literals is as hard as in propositional logic

Solution: Exploit propositional satisfiability technology

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Lifting SAT Technology to SMT

Two main approaches:

1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]
   - translate into an equisatisfiable propositional formula
   - feed it to any SAT solver

Notable systems: *UCLID*

2. "Lazy" [ACG00, dMR02, BDS02, ABC+02]
   - abstract the input formula to a propositional one
   - feed it to a (DPLL-based) SAT solver
   - use a theory decision procedure to refine the formula and guide the SAT solver

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(Very) Lazy Approach for SMT – Example

$$g(a) = c \quad \wedge \quad f(g(a)) \neq f(c) \;\vee\; g(a) = d \quad \wedge \quad c \neq d$$

Theory T**:** Equality with Uninterpreted Functions

Simplest setting:

- Off-line SAT solver
- Non-incremental *theory solver* for conjunctions of equalities and disequalities
- Theory atoms (e.g., $g(a) = c$) abstracted to propositional atoms (e.g., 1)

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Lazy Approach – Enhancements

Several enhancements are possible to increase efficiency:

- Check T-satisfiability only of full propositional model
- Check T-satisfiability of partial assignment M as it grows
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- If M is T-unsatisfiable, identify a T-unsatisfiable subset M_0 of M and add $\neg M_0$ as a clause
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Lazy Approach – Main Benefits

- Every tool does what it is good at:
    - SAT solver takes care of Boolean information
    - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
    - SAT and theory solvers communicate via a simple API [GHN+04]
    - SMT for a new theory only requires new theory solver
    - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)

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An Abstract Framework for Lazy SMT

Several variants and enhancements of lazy SMT solvers exist

They can be modeled abstractly and declaratively as *transition systems*

A transition system is a binary relation over states, induced by a set of conditional transition rules

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07]

Advantages of Abstract Framework

An abstract framework helps one:

- skip over implementation details and unimportant control aspects
- reason formally about solvers for SAT and SMT
- model advanced features such as non-chronological bactracking, lemma learning, theory propagation, …
- describe different strategies and prove their correctness
- compare different systems at a higher level
- get new insights for further enhancements

The one described next is a re-elaboration of those in [NOT06, KG07]

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The Original DPLL Procedure

- Modern SAT solvers are based on the DPLL procedure [DP60, DLL62]
- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F
- M is grown by
  - deducing the truth value of a literal from M and F, or
  - guessing a truth value
- If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

An Abstract Framework for DPLL

States:

$$\mathsf{fail} \quad \text{or} \quad \langle M, F \rangle$$

where

- M is a sequence of literals and *decision points* $\bullet$ denoting a partial truth *assignment*
- F is a set of clauses denoting a CNF *formula*

Def. If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level* i of M
- $M^{[i]} \stackrel{\text{def}}{=} M_0 \bullet \cdots \bullet M_i$

An Abstract Framework for DPLL

States:

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Initial state:

- $\langle (), F_0 \rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- fail if F_0 is unsatisfiable
- $\langle M, G \rangle$ otherwise, where
  - G is equivalent to F_0 and
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Transition Rules: Notation

States treated like records:

- M denotes the truth assignment component of current state
- F denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

$$\frac{p_1 \quad \cdots \quad p_n}{[\mathsf{M} := e_1] \quad [\mathsf{F} := e_2]}$$

updating M, F or both when premises $p_1, \ldots, p_n$ all hold

Transition Rules for the Original DPLL

Extending the assignment

Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\; l}$$

Note: When convenient, treat M as a set

Note: Clauses are treated modulo ACI of $\vee$

Decide $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

Note: $\mathrm{Lit}(F) \stackrel{\text{def}}{=} \{l \mid l \text{ literal of } F\} \cup \{\bar{l} \mid l \text{ literal of } F\}$

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Transition Rules for the Original DPLL

Repairing the assignment

Fail
$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\, N \quad \bullet \notin N}{\mathsf{M} := M\, \bar{l}}$$

Note: Last premise of **Backtrack** enforces chronological backtracking

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From DPLL to CDCL Solvers (1)

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

States: fail or $\langle M, F, C \rangle$

Initial state:

- $\langle (), F_0, \mathsf{no} \rangle$, where F_0 is to be checked for satisfiability

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From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

Conflict $\dfrac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$

Explain $\dfrac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$

Backjump $\dfrac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\, \bar{l}_1, \ldots, \mathsf{lev}\, \bar{l}_n \leqslant i < \mathsf{lev}\, \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\, l}$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg \mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Note: $\models_{\mathrm{p}}$ denotes propositional entailment

From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

$$\textbf{Conflict}\ \frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

$$\textbf{Explain}\ \frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

$$\textbf{Backjump}\ \frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\ \bar{l}_1, \ldots, \mathsf{lev}\ \bar{l}_n \leqslant\ i < \mathsf{lev}\ \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\ l}$$

Note: $l \prec_{\mathsf{M}} l'$ if l occurs before l' in M
$\mathsf{lev}\ l = i$ iff l occurs in decision level i of M

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

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From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

$$\textbf{Conflict}\ \frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

$$\textbf{Explain}\ \frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

$$\textbf{Backjump}\ \frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\, \bar{l}_1, \ldots, \mathsf{lev}\, \bar{l}_n \leqslant\ i < \mathsf{lev}\, \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\, l}$$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Note: $\models_{\mathrm{p}}$ denotes propositional entailment

From DPLL to CDCL Solvers (3)

Modify **Fail** to

$$\textbf{Fail} \quad \frac{\mathsf{C} \neq \mathsf{no} \qquad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

From DPLL to CDCL Solvers (3)

Modify **Fail** to

$$\textbf{Fail}\ \frac{\mathsf{C} \neq \mathsf{no} \qquad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \lor 2,\ \ \overline{3} \lor 4,\ \ \overline{5} \lor \overline{6},\ \ \overline{1} \lor \overline{5} \lor 7,\ \ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \lor \overline{5} \lor 6 \lor \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \lor \overline{2} \lor \overline{5} \lor 6$	by **Explain** with $\overline{1} \lor \overline{5} \lor 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
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Execution Example

$$F := \{1,\ \ \overline{1} \lor 2,\ \ \overline{3} \lor 4,\ \ \overline{5} \lor \overline{6},\ \ \overline{1} \lor \overline{5} \lor 7,\ \ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \lor \overline{5} \lor 6 \lor \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \lor \overline{2} \lor \overline{5} \lor 6$	by **Explain** with $\overline{1} \lor \overline{5} \lor 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$
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$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ \overline{7}$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
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Execution Example

$$F := \{1,\ \ \overline{1} \lor 2,\ \ \overline{3} \lor 4,\ \ \overline{5} \lor \overline{6},\ \ \overline{1} \lor \overline{5} \lor 7,\ \ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

From DPLL to CDCL Solvers (4)

Also add

Learn $$\frac{\mathsf{F} \models_{\mathrm{p}} C \quad C \notin \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{C\}}$$

Forget $$\frac{\mathsf{C} = \mathsf{no} \quad \mathsf{F} = G \cup \{C\} \quad G \models_{\mathrm{p}} C}{\mathsf{F} := G}$$

Restart $$\frac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad \mathsf{C} := \mathsf{no}}$$

Note: Learn can be applied to any clause stored in C when $\mathsf{C} \neq \mathsf{no}$

Modeling Modern SAT Solvers

At the core, current CDCL SAT solvers are implementations of the transition system with rules

Propagate, **Decide**,

Conflict, **Explain**, **Backjump**,

Learn, **Forget**, **Restart**

Basic DPLL $\stackrel{\text{def}}{=}$

{ **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump** }

DPLL $\stackrel{\text{def}}{=}$ Basic DPLL + { **Learn**, **Forget**, **Restart** }

Modeling Modern SAT Solvers

At the core, current CDCL SAT solvers are implementations of the transition system with rules

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Conflict, **Explain**, **Backjump**,

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Basic DPLL $\stackrel{\text{def}}{=}$

{ **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump** }

DPLL $\stackrel{\text{def}}{=}$ Basic DPLL + { **Learn**, **Forget**, **Restart** }

The Basic DPLL System – Correctness

Some terminology:

Irreducible state: state for which no Basic DPLL rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, the clause set F_0 is satisfied by M.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Note: This is not so immediate, because of **Backjump**.

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

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The DPLL System – Strategies

- Applying
  - one Basic DPLL rule between each two **Learn** applications and
  - **Restart** less and less often

  ensures termination

- A common basic strategy applies the rules with the following priorities:
  1. If $n > 0$ conflicts have been found so far, increase n and apply **Restart**
  2. If a clause is falsified by M, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
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  6. Apply **Propagate** to completion
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From SAT to SMT

Same states and transitions but

- F contains quantifier-free clauses in some theory T
- M is a sequence of theory literals and decision points
- the DPLL system is augmented with rules

  T**-Conflict**, T**-Propagate**, T**-Explain**
- maintains invariant: $\mathsf{F} \models_T \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Def. $F \models_T G$ iff every model of T that satisfies F satisfies G as well

SMT-level Rules

Fix a theory T

$$T\textbf{-Conflict}\ \frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

$$T\textbf{-Propagate}\ \frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

$$T\textbf{-Explain}\ \frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

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T-Propagate $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$$

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T**-Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier UF example

$$\underbrace{g(a)=c}_{1} \quad \wedge \quad \underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3} \quad \wedge \quad \underbrace{c\neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	$\overline{1}\vee 2\vee 4$	by T**-Conflict**
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	$\overline{1}\vee 2\vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4,\ \overline{1}\vee\overline{3}\vee 4$	$\overline{1}\vee\overline{3}\vee 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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fail			by **Fail**

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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

A Better Lazy Approach

The very lazy approach can be improved considerably with

- An *on-line* SAT engine,
  which can accept new input clauses on the fly
- an *incremental and explicating T*-solver,
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  1. check the T-satisfiability of M as it is extended and
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Lazy Approach – Strategies

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
2. If M is T-unsatisfiable, apply T**-Conflict**
3. Apply **Fail** or **Explain+Learn+Backjump** as appropriate
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Theory Propagation

With T**-Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

With T**-Propagate** and T**-Explain**, it can also be used to guide the engine's search [Tin02]

$$T\textbf{-Propagate} \quad \frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

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no applications of **Decide** are needed

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Modeling Modern Lazy SMT Solvers

At the core, current lazy SMT solvers are implementations of the transition system with rules

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) T**-Conflict**, T**-Propagate**, T**-Explain**

(3) **Learn**, **Forget**, **Restart**

Basic DPLL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2)

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Correctness

Updated terminology:

Irreducible state: state to which no Basic DPLL MT rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is T-unsatisfiable.

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Correctness

Updated terminology:

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Exhausted execution: execution ending in an irreducible state

Proposition (Termination) Every execution in which

(a) **Learn/Forget** are applied only finitely many times and

(b) **Restart** is applied with increased periodicity

is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

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DPLL(T) Architecture

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

DPLL(X):

- Very similar to a SAT solver, enumerates Boolean models
- Not allowed: pure literal, blocked literal detection, ...
- Required: incremental addition of clauses
- Desirable: partial model detection

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

T-solver:

- Checks the T-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of T-unsatisfiability/propagation
- Must be incremental and backtrackable

Reasoning by Cases in Theory Solvers

For certain theories, determining that a set M is T-unsatisfiable requires reasoning by cases.

Example: T = the theory of arrays.

$$M = \{\underbrace{r(w(a,i,x),j) \neq x}_{1},\ \underbrace{r(w(a,i,x),j) \neq r(a,j)}_{2}\}$$

$i = j$) Then, $r(w(a,i,x),j) = x$. Contradiction with 1.

$i \neq j$) Then, $r(w(a,i,x),j) = r(a,j)$. Contradiction with 2.

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Case Splitting

A *complete* T-solver reasons by cases via (internal) case splitting and backtracking mechanisms

An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine

Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them [BNOT06]

Possible benefits:

- All case-splitting is coordinated by the SAT engine
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Basic Scenario:

$$\mathsf{M} = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

- Main SMT module: "Is M T-unsatisfiable?"
- T-solver: "I do not know yet, but it will help me if you consider these *theory lemmas*:

$$s = s' \wedge i = j \rightarrow s = t, \quad s = s' \wedge i \neq j \rightarrow s = r(a,j)\text{"}$$

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Modeling Splitting on Demand

To model the generation of theory lemmas for case splits, add the rule

T-Learn

$$\frac{\models_T \exists \mathbf{v}(l_1 \vee \cdots \vee l_n) \quad l_1, \ldots, l_n \in L_{\mathrm{S}} \quad \mathbf{v} \text{ vars not in } \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

Note: For many theories with a theory solver, there exists an appropriate finite L_{S} for every input F

The set L_{S} does not need to be computed explicitly

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Now we can relax the requirement on the theory solver:

When $\mathsf{M} \models_{\mathrm{p}} \mathsf{F}$, *it must either*

- *determine whether* $\mathsf{M} \models_T \perp$ *or*
- *generate a new clause by* T***-Learn*** *containing at least one literal of* L_{S} *undefined in* M

The T-solver is required to determine whether $\mathsf{M} \models_T \perp$ only if all literals in L_{S} are defined in M

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Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \bullet y = \emptyset$	F	by **Decide**
$x = y \cup z \bullet y = \emptyset\ x \neq z$	F	by **Propagate**
$x = y \cup z \bullet y = \emptyset\ x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by $\mathcal{T}$**-Learn**
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$\mathcal{T}$-solver can make the following deductions at this point:

$$e \in x \quad \cdots \Rightarrow \quad e \in y \cup z \quad \cdots \Rightarrow \quad e \in y \quad \cdots \Rightarrow \quad e \in \emptyset \quad \Rightarrow \perp$$

This enables an application of $\mathcal{T}$**-Conflict** with clause

$$x \neq y \cup z \ \vee \ y \neq \emptyset \ \vee \ x = z \ \vee \ e \notin x \ \vee \ e \in z$$

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$x = y \cup z \ \bullet \ y = \emptyset\ x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by T**-Learn**
$x = y \cup z \ \bullet \ y = \emptyset\ x \neq z \ \bullet \ e \in x$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Decide**
$x = y \cup z \ \bullet \ y = \emptyset\ x \neq z \ \bullet \ e \in x\ e \notin z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Propagate**

T-solver can make the following deductions at this point:

$$e \in x \quad \cdots \quad \Rightarrow \ e \in y \cup z \quad \cdots \quad \Rightarrow \ e \in y \quad \cdots \quad \Rightarrow \ e \in \emptyset \ \Rightarrow \ \perp$$

This enables an application of T**-Conflict** with clause

$$x \neq y \cup z \ \vee \ y \neq \emptyset \ \vee \ x = z \ \vee \ e \notin x \ \vee \ e \in z$$

Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \bullet y = \emptyset$	F	by **Decide**
$x = y \cup z \bullet y = \emptyset\, x \neq z$	F	by **Propagate**
$x = y \cup z \bullet y = \emptyset\, x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **T-Learn**
$x = y \cup z \bullet y = \emptyset\, x \neq z \bullet e \in x$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Decide**
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Correctness Results

Correctness results can be extended to the new rule.

Soundness: The new T**-Learn** rule maintains satisfiability of the clause set.

Completeness: As long as the theory solver can decide $\mathsf{M} \models_T \bot$ when all literals in L_{S} are determined, the system is still complete.

Termination: The system terminates under the same conditions as before. Roughly:

- Any lemma is (re)learned only finitely many times
- **Restart** is applied with increased periodicity

Combining Theories

Need for Combining Theories and Solvers

Recall: Many applications give rise to formulas like:

$$a \approx b + 2 \;\wedge\; A \approx \mathrm{write}(B, a + 1, 4) \;\wedge$$
$$(\mathrm{read}(A, b + 3) \approx 2 \;\vee\; f(a - 1) \not\approx f(b + 1))$$

Solving that formula requires reasoning over

- the theory of linear arithmetic (T_{LA})
- the theory of arrays (T_{A})
- the theory of uninterpreted functions (T_{UF})

Question: Given solvers for each theory, can we combine them modularly into one for $T_{\mathrm{LA}} \cup T_{\mathrm{A}} \cup T_{\mathrm{UF}}$?

Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]

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Motivating Example (Convex Case)

Consider the following set of literals over $T_{\mathrm{LRA}} \cup T_{\mathrm{UF}}$ (T_{LRA}, linear real arithmetic):

$$\begin{aligned} f(f(x) - f(y)) &= a \\ f(0) &> a + 2 \\ x &= y \end{aligned}$$

First step: *purify* literals so that each belongs to a single theory

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$$f(f(x) - f(y)) = a \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = f(x) - f(y) \end{array} \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = e_2 - e_3 \\ e_2 = f(x) \\ e_3 = f(y) \end{array}$$

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$$f(0) > a + 2 \quad \Longrightarrow \quad \begin{array}{c} f(e_4) > a + 2 \\ e_4 = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} f(e_4) = e_5 \\ e_4 = 0 \\ e_5 > a + 2 \end{array}$$

Motivating Example (Convex Case)

Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

L_1	L_2
$f(e_1) = a$	$e_2 - e_3 = e_1$
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$f(y) = e_3$	$e_5 > a + 2$
$f(e_4) = e_5$	$e_2 = e_3$
$x = y$	$a = e_5$
$e_1 = e_4$	

$L_1 \models_{\text{UF}} e_2 = e_3$ $\quad$ $L_2 \models_{\text{LRA}} e_1 = e_4$

$L_1 \models_{\text{UF}} a = e_5$

Third step: check for satisfiability locally

$L_2 \models_{\text{LRA}} \bot$ $\quad$ Report unsatisfiable

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Third step: check for satisfiability locally

$L_1 \models_{\mathrm{UF}} \bot$
$L_2 \models_{\mathrm{LRA}} \bot$ $\qquad$ Report unsatisfiable

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$L_1 \models_{\mathrm{UF}} \bot$
$L_2 \models_{\mathrm{LRA}} \bot$

Report unsatisfiable

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$L_1 \models_{\mathrm{UF}} \perp$
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$L_1 \not\models_{\mathrm{UF}} \bot$

$L_2 \models_{\mathrm{LRA}} \bot$

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$L_1 \not\models_{\mathrm{UF}} \bot$

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Report unsatisfiable

Motivating Example (Non-convex Case)

Consider the following unsatisfiable set of literals over $T_{\mathrm{LIA}} \cup T_{\mathrm{UF}}$ (T_{LIA}, linear integer arithmetic):

$$\begin{aligned} 1 \leqslant \quad x \quad &\leqslant 2 \\ f(1) &= a \\ f(2) &= f(1) + 3 \\ a &= b + 2 \end{aligned}$$

First step: *purify* literals so that each belongs to a single theory

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$$f(1) = a \quad \Longrightarrow \quad \begin{aligned} f(e_1) &= a \\ e_1 &= 1 \end{aligned}$$

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First step: *purify* literals so that each belongs to a single theory

$$f(2) = f(1) + 3 \quad \Longrightarrow \quad \begin{aligned} e_2 &= 2 \\ f(e_2) &= e_3 \\ f(e_1) &= e_4 \\ e_3 &= e_4 + 3 \end{aligned}$$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leqslant x$	$f(e_1) = a$
$x \leqslant 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_1$
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No more entailed equalities, but $L_1 \models_{\text{LIA}} x = e_1 \vee x = e_2$

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$e_2 = 2$	$x = e_1$
$e_3 = e_4 + 3$	
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Consider each case of $x = e_1 \vee x = e_2$ separately

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Case 1) $x = e_1$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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$L_2 \models_{\mathrm{UF}} a = b$, which entails $\perp$ when sent to L_1

Motivating Example (Non-convex Case)

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$x = e_2$	

Case 2) $x = e_2$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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$L_2 \models_{\mathrm{UF}} e_3 = b$, which entails $\perp$ when sent to L_1

The Nelson-Oppen Method

- For $i = 1, 2$, let T_i be a first-order theory of *signature* Σ_i (set of function and predicate symbols in T_i other than $=$)
- Let $T = T_1 \cup T_2$
- Let $\mathcal{C}$ be a finite set of *free* constants (i.e., not in $\Sigma_1 \cup \Sigma_2$)

We consider only input problems of the form

$$L_1 \cup L_2$$

where each L_i is a finite set of *ground* (i.e., variable-free) $(\Sigma_i \cup \mathcal{C})$-literals

Note: Because of purification, there is no loss of generality in considering only ground $(\Sigma_i \cup \mathcal{C})$-literals

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The Nelson-Oppen Method

- For $i = 1, 2$, let T_i be a first-order theory of *signature* Σ_i (set of function and predicate symbols in T_i other than $=$)
- Let $T = T_1 \cup T_2$
- Let $\mathcal{C}$ be a finite set of *free* constants (i.e., not in $\Sigma_1 \cup \Sigma_2$)

We consider only input problems of the form

$$L_1 \cup L_2$$

where each L_i is a finite set of *ground* (i.e., variable-free) $(\Sigma_i \cup \mathcal{C})$-literals

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The Nelson-Oppen Method

Bare-bones, non-deterministic, non-incremental version

[Opp80, Rin96, TH96]:

Input: $L_1 \cup L_2$ with L_i finite set of ground $(\Sigma_i \cup \mathcal{C})$-literals
Output: **sat** or **unsat**

1. Guess an *arrangement* A, i.e., a set of equalities and disequalities over $\mathcal{C}$ such that

$$c = d \in A \;\text{ or }\; c \neq d \in A \;\text{ for all } c, d \in \mathcal{C}$$

2. If $L_i \cup A$ is T_i-unsatisfiable for $i = 1$ or $i = 2$, return unsat
3. Otherwise, return **sat**

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Correctness of the NO Method

Proposition (Termination) The method is terminating.

(Trivially, because there is only a finite number of arrangements to guess)

Proposition (Soundness) If the method returns **unsat** for every arrangement, the input is $(T_1 \cup T_2)$-unsatisfiable.

(Because satisfiability in $(T_1 \cup T_2)$ is always preserved)

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Stably Infinite Theories

Def. A theory T is *stably infinite* iff every quantifier-free T-satisfiable formula is satisfiable in an infinite model of T

Many interesting theories are stably infinite:

- Theories of an infinite structure (e.g., integer arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
- Convex theories (e.g., EUF, linear real arithmetic)

Def. A theory T is *convex* iff, for any set L of literals
$L \models_T s_1 = t_1 \vee \cdots \vee s_n = t_n \implies L \models_T s_i = t_i$ for some i

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SMT Solving with *Multiple* Theories

Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$

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Quick Solution:

1. Combine $S_1, \dots, S_n$ with Nelson-Oppen into a theory solver for T
2. Build a DPLL(T) solver as usual

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Better Solution [Bar02, BBC+05b, BNOT06]:

1. Extend DPLL(T) to DPLL($T_1, \ldots, T_n$)
2. Lift Nelson-Oppen to the DPLL($X_1, \ldots, X_n$) level
3. Build a DPLL($T_1, \ldots, T_n$) solver

Modeling DPLL($T_1, \ldots, T_n$) Abstractly

- Let $n = 2$, for simplicity
- Let T_i be of signature Σ_i for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$
- Let $\mathcal{C}$ be a set of free constants
- Assume wlog that each input literal has signature $(\Sigma_1 \cup \mathcal{C})$ or $(\Sigma_2 \cup \mathcal{C})$ (no *mixed* literals)
- Let $\mathsf{M}|_i \overset{\text{def}}{=} \{(\Sigma_i \cup \mathcal{C})\text{-literals of } \mathsf{M} \text{ and their complement}\}$
- Let $\mathsf{I}(\mathsf{M}) \overset{\text{def}}{=} \{c = d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\} \cup$
  $\{c \neq d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\}$

(*interface literals*)

Abstract DPLL Modulo Multiple Theories

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

Decide $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$

Only change: decide on interface equalities as well

***T*-Propagate** $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\} \quad \mathsf{M} \models_{T_i} l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$

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$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

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I-Learn

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

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$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$
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Example — Non-convex Theories

$$F := \begin{array}{l} \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge \\ \underbrace{1 \leqslant x}_{4} \wedge \underbrace{x \leqslant 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9} \end{array}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I**-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **l-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\ldots$			(exercise)
fail	$\ldots$	$\ldots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\ldots$			(exercise)
fail	$\ldots$	$\ldots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by ***T*-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by ***T*-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by ***T*-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by ***T*-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ \overline{3} \models_{UF} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{LIA} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{UF} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{UF} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

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M	F	C	rule
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$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by I-**Learn** $(\models_{\mathrm{LIA}} \overline{4} \lor \overline{5} \lor 11 \lor 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	$\overline{7} \lor \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Propagate**
$\cdots$			(exercise)
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$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{UF} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{LIA} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{UF} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{UF} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^{+}$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ \overline{3} \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by I-**Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ \underline{3} \models_{UF} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by I-**Learn** $(\models_{LIA} \overline{4} \lor \overline{5} \lor 11 \lor 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{UF} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	$\overline{7} \lor \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{UF} \perp)$
$0 \cdots 9\ \underline{10}\ \underline{\overline{13}}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \begin{array}{l}\overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge \\ \underbrace{1 \leqslant x}_{4} \wedge \underbrace{x \leqslant 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}\end{array}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ \underline{3} \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by I**-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{\overline{13}}\ \overline{11}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
$$\underbrace{1 \leqslant x}_{4} \wedge \underbrace{x \leqslant 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I-**Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
$$\underbrace{1 \leqslant x}_{4} \wedge \underbrace{x \leqslant 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I-**Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\ldots$			(exercise)
fail	$\ldots$	$\ldots$	by **Fail**

Suggested Readings

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